# SPECTRAL BEHAVIOUR OF
# A SIMPLE NON-SELF-ADJOINT OPERATOR

LYONELL S. BOULTON[†]

Department of Mathematics, King's College London
The Strand, London WC2R 2LS, U.K.
email: lboulton@mth.kcl.ac.uk

ABSTRACT. We investigate the spectrum of a typical non-self-adjoint differential operator $AD = -d^2/dx^2 \otimes A$ acting on $L^2(0,1) \otimes \mathbb{C}^2$, where $A$ is a $2 \times 2$ constant matrix. We impose Dirichlet and Neumann boundary conditions in the first and second coordinate respectively at both ends of $[0,1] \subset \mathbb{R}$. For $A \in \mathbb{R}^{2\times 2}$ we explore in detail the connection between the entries of $A$ and the spectrum of $AD$, we find necessary conditions to ensure similarity to a self-adjoint operator and give numerical evidence that suggests a non-trivial spectral evolution.

## 1. INTRODUCTION

In this paper we investigate spectral properties of the linear operator $AD$ acting on $L^2(0,1) \otimes \mathbb{C}^2$ where $A$ is a $2 \times 2$ constant matrix and $D$ denotes the ordinary differential operator

$$D \begin{pmatrix} \phi \\ \gamma \end{pmatrix} := - \begin{pmatrix} \phi'' \\ \gamma'' \end{pmatrix}, \qquad \begin{matrix} \phi(0) = \phi(1) = 0 \\ \gamma'(0) = \gamma'(1) = 0. \end{matrix}$$

The apparently simple combination of Dirichlet and Neumann boundary conditions allows self-adjointness if, an only if, $A$ is real and diagonal. If $A$ is non-diagonal and upper-triangular the numerical range of $AD$ is a large sector of $\mathbb{C}$. Otherwise it is the whole of $\mathbb{C}$ preventing us from applying the theory of sectorial sesquilinear forms in a straightforward manner.

Our main goal is to explore the connection between the entries of the matrix $A$ and the location of the spectrum of $AD$ in the complex plane. In [8] R. F. Streater considers the particular case

$$A = \begin{pmatrix} 1 & \gamma \\ 1/2\gamma & 1 \end{pmatrix}, \qquad \gamma > 0,$$

in order to find necessary conditions for the stability of small perturbations about the stationary solution of certain non-linear system of parabolic equations. Streater's system represents a thermodynamical

model for hot fluid in one dimension and the localization of the spectrum is achieved by constructing a non-unitary transformation that makes $AD$ similar to a non-negative self-adjoint operator, hence the spectrum of $AD$ is real and non-negative. This similarity transformation does not work for other matrices and a slight modification of the entries of $A$ can destroy reality of the spectrum (cf. sections 6-7) so the general case should be attacked by other methods.

Although this paper mainly concerns $A \in \mathbb{R}^{2 \times 2}$, the results of sections 2-5 refer to any complex $2 \times 2$ matrix. The core results are to be found in section 6 where we present an exhaustive description of the spectrum of $AD$ in terms of the entries of $A$. Among various other unexpected conclusions, the following three epitomize the complexity of the problem to be considered:

a) When $A$ is triangular and non-diagonalizable, $AD$ is not similar to a self-adjoint operator but the spectrum of $AD$ is real (theorem 20).
b) The spectrum of $AD$ can be non-real even when both eigenvalues of $A$ are positive and equal (theorem 27).
c) There is a continuous family of matrices $A$ whose eigenvalues do not intersect the real line but such that the spectrum of $AD$ is real (theorem 24).

The last two assertions show that the spectra of $A$, $D$ and $AD$ are typically unrelated.

The crucial idea in section 6 is to reduce the four-parameter problem of localizing the spectrum of $AD$ in terms of the entries of $A$, to five two-parameter cases and describe separately each of these cases. Sections 2-5 are devoted to describing the various properties of $AD$ we will use in section 6, whereas section 7 is devoted to numerical computations which illustrate some of the results reported. In section 2 we find the boundary conditions associated to the adjoint of $AD$ and compute the numerical range of $AD$. In section 3 we show that the resolvent of $AD$ is compact for all non-singular $A$. In Section 4 we explore the stability of the spectrum of $AD$ in the sense of [4] and [9], and provide estimates which allow us to enclose the spectrum of $AD$ in angular regions when $A$ is subject to various constraints. In section 5 we use standard ODE methods to compute the transcendental function of the spectral problem associated to $AD$.

## 2. Definitions and notation

Let $K$ be a linear operator whose domain is denoted by $\mathrm{Dom}(K)$. Throughout this paper $\mathrm{Spec}(K)$ stands for the spectrum of $K$ and the numerical range of $K$ is defined to be

$$\mathrm{Num}\,(K) := \{\langle Kf, f \rangle : f \in \mathrm{Dom}(K), \|f\| = 1\}.$$

We recall that the numerical range of any linear operator is convex and that if $\mathrm{Spec}\,(K) \neq \emptyset$, then

$$\mathrm{Spec}\,(K) \subset \overline{\mathrm{Num}\,(K)}.$$

If $K = K^*$ and $\mathrm{Spec}\,(K) \subset (0, \infty)$, we will say that $K$ is positive and write $K > 0$. If $K = K^*$ and $\mathrm{Spec}\,(K) \subset [0, \infty)$, we will say that $K$ is non-negative and write $K \geq 0$.

Below and elsewhere $|v|$ denotes the norm of a vector $v \in \mathbb{C}^2$. The norm of any

$$f \equiv \begin{pmatrix} \phi \\ \gamma \end{pmatrix} \in \mathrm{L}^2(0, 1) \otimes \mathbb{C}^2$$

is the standard Hilbert tensor product norm

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 |f(x)|^2 \mathrm{d}x = \int_0^1 \left( |\phi(x)|^2 + |\gamma(x)|^2 \right) \mathrm{d}x.$$

Unless explicitly stated, we denote

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The complex numbers $a_+, a_-$ denote the eigenvalues of $A$ and the non-zero $\mathbb{C}^2$ vectors $v_+, v_-$ denote the eigenvectors

$$Av_{\pm} = a_{\pm}v_{\pm}.$$

If $a_+$ and $a_-$ are real and different, we adopt the convention $a_- < a_+$. Notice that the $v_{\pm}$ are not necessarily orthogonal.

Let $\mathrm{W}^{2,2}$ be the Sobolev space of all $f \in \mathrm{L}^2(0, 1) \otimes \mathbb{C}^2$, such that the generalized derivative $f'' \in \mathrm{L}^2(0, 1) \otimes \mathbb{C}^2$. We define rigorously the domain of $AD$ as

$$\mathrm{Dom}(D) = \{f \in \mathrm{W}^{2,2} \; : \; \phi(0) = \phi(1) = 0, \; \gamma'(0) = \gamma'(1) = 0\}.$$

If $A$ is invertible, it is standard to show that $AD$ is always a closed densely defined linear operator acting on $\mathrm{L}^2(0, 1) \otimes \mathbb{C}^2$.

**Lemma 1.** *If $A$ is singular, then $AD$ is not closed in the domain* $\mathrm{Dom}(D)$.

**Proof.** Let $v \in \mathbb{C}^2$ be a non vanishing vector such that $Av = 0$ and let $f(x) := vx \in \mathrm{L}^2(0, 1) \otimes \mathbb{C}^2$. Clearly $f \notin \mathrm{Dom}(D)$. Let $\phi_n$ be a sequence of smooth functions whose support is compact in $(0, 1)$ and such that $\phi_n(x) \to x$ in $\mathrm{L}^2(0, 1)$. Then $\phi_n v \in \mathrm{Dom}(D)$ and $\phi_n v \to f$. Also

$$AD\phi_n(x)v = -\phi_n''(x)Av = 0,$$

so that $AD(\phi_n v)$ is a convergent sequence in $\mathrm{L}^2(0, 1) \otimes \mathbb{C}^2$. We complete the proof by noticing that if $AD$ was closed, then we would have $f \in \mathrm{Dom}(D)$. ∎

For the rest of this section and in sections 3-5 we will assume without further mention that $A$ is non-singular. In section 6 we will consider again singular $A$.

In order to show that $AD$ is in general non-self-adjoint, let us compute the adjoint $(AD)^*$. Let

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the boundary conditions for $D$ can be rewritten as

$$Pf(0) = Pf(1) = 0, \qquad (I - P)f'(0) = (I - P)f'(1) = 0.$$

**Lemma 2.** *The adjoint of $AD$ is*

$$(AD)^* f = -A^* f'',$$

*for $f \in \mathrm{W}^{2,2}$ subject to the boundary conditions*

$$\hat{P}f(0) = \hat{P}f(1) = 0$$
$$(I - \hat{P})f'(0) = (I - \hat{P})f'(1) = 0 \tag{1}$$

*where $\hat{P} = \hat{P}^2$ is the rank one projection such that*

$$\mathrm{Ran}\left(\hat{P}\right) = \mathrm{Ran}\left(A(I - P)\right)^{\perp}$$
$$\mathrm{Ran}\left(I - \hat{P}\right) = \mathrm{Ran}\left(AP\right)^{\perp}. \tag{2}$$

**<u>Proof.</u>** For $f \in \mathrm{Dom}(D)$ and $g \in \mathrm{L}^2(0,1) \otimes \mathbb{C}^2$,

$$
\begin{aligned}
\langle ADf, g \rangle &= -\int_0^1 \langle Af''(x), g(x) \rangle \, \mathrm{d}x \\
&= \langle APf', g \rangle|_1^0 + \int_0^1 \langle Af'(x), g'(x) \rangle \, \mathrm{d}x.
\end{aligned}
$$

We ought to find a complex $2 \times 2$ matrix $B$ and impose boundary conditions on $g$, for

$$
\begin{aligned}
\langle f, (AD)^* g \rangle &= -\int_0^1 \langle f(x), Bg''(x) \rangle \, \mathrm{d}x \\
&= \langle B^* f, g' \rangle|_1^0 + \int_0^1 \langle B^* f'(x), g'(x) \rangle \, \mathrm{d}x \\
&= \langle B^*(I - P)f, g' \rangle|_1^0 + \int_0^1 \langle B^* f'(x), g'(x) \rangle \, \mathrm{d}x.
\end{aligned}
$$

and

$$\langle ADf, g \rangle = \langle f, (AD)^* g \rangle.$$

This must be true in particular for all $f$ and $g$ with compact support in $(0, 1)$, so clearly $B = A^*$.

Let the boundary conditions for $(AD)^*$ be given by (1) where $\hat{P} = \hat{P}^2$ is a non-necessarily orthogonal projection on $\mathbb{C}^2$, we show (2). If $f, g$ are smooth functions supported in $[0, 1)$, then

$$\left\langle APf'(0), (I - \hat{P})g(0) \right\rangle = \left\langle A(I - P)f(0), \hat{P}g'(0) \right\rangle$$

where $f(0)$, $f'(0)$, $g(0)$ and $g'(0)$ are arbitrary vectors in $\mathbb{C}^2$. If $f'(0) = 0$, the right hand side should vanish for all $f(0), g'(0) \in \mathbb{C}^2$, so that

$$\mathrm{Ran}\left(\hat{P}\right) = \mathrm{Ran}\left(A(I - P)\right)^{\perp}.$$

If $f(0) = 0$, the left hand side should vanish for all $f'(0), g(0) \in \mathbb{C}^2$, so that

$$\mathrm{Ran}\left(I - \hat{P}\right) = \mathrm{Ran}\left(AP\right)^{\perp}.$$

Since $A$ is non-singular these two spaces are one dimensional. ∎

**Corollary 3.** *$AD$ is self-adjoint, if and only if $A$ is real and diagonal.*

**Proof.** Using the notation of lemma 2, $AD$ is self-adjoint, if and only if

$$A = A^* \qquad \text{and} \qquad P = \hat{P}.$$

The latter occurs, if and only if

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

These conditions ensure $A$ real and diagonal. ∎

We now show that due to the boundary conditions we have chosen,

$$\mathrm{Num}\,(AD) = \mathbb{C}$$

for a large family of non-diagonal matrices $A$. This prevents us from employing the theory of sectorial sesquilinear forms in order to find the spectrum.

**Theorem 4.** *Let $A$ be a non-singular matrix.*

a) *If $A$ is an upper triangular matrix (that is $c = 0$), then*

$$\overline{\mathrm{Num}\,(AD)} = \{rz : r \in [0, \infty), z \in \overline{\mathrm{Num}\,(A)}\}.$$

b) *If $A$ is not an upper triangular matrix (that is $c \neq 0$), then*

$$\mathrm{Num}\,(AD) = \mathbb{C}.$$

**Proof.** Since $0$ is always an eigenvalue of $AD$ (cf. section 3), then $0 \in \mathrm{Num}\,(AD)$. For $f \in \mathrm{Dom}(D)$,

$$
\begin{aligned}
\langle ADf, f \rangle &= -\int_0^1 \langle Af''(x), f(x) \rangle \, \mathrm{d}x \\
&= \langle Af', f \rangle |_1^0 + \int_0^1 \langle Af'(x), f'(x) \rangle \, \mathrm{d}x \\
&= \left\langle A \begin{pmatrix} \phi' \\ \gamma' \end{pmatrix}, \begin{pmatrix} \phi \\ \gamma \end{pmatrix} \right\rangle \Big|_1^0 + \int_0^1 \langle Af'(x), f'(x) \rangle \, \mathrm{d}x \\
&= \left\langle A \begin{pmatrix} \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \right\rangle \Big|_1^0 + \int_0^1 \langle Af'(x), f'(x) \rangle \, \mathrm{d}x. \quad (3)
\end{aligned}
$$

Case a): call

$$
\Phi := \{ rz : r \in [0, \infty), z \in \overline{\mathrm{Num}\,(A)} \}.
$$

Then $\Phi$ is a convex set and

$$
\begin{aligned}
\Phi &= \overline{\{ rz : r \in [0, \infty), z \in \mathrm{Num}\,(A) \}} \\
&= \overline{\{ \langle Av, v \rangle : v \in \mathbb{C}^2 \}}.
\end{aligned}
$$

If $c = 0$,

$$
\left\langle A \begin{pmatrix} \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \right\rangle \Big|_1^0 = \left\langle \begin{pmatrix} a\phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \right\rangle \Big|_1^0 = 0
$$

so that

$$
\langle ADf, f \rangle = \int_0^1 \langle Af'(x), f'(x) \rangle \, \mathrm{d}x.
$$

This and the fact that $\Phi$ is closed and convex, yield

$$
\overline{\mathrm{Num}\,(AD)} \subseteq \Phi.
$$

In order to prove the reverse inclusion, let $v \in \mathbb{C}^2$ be such that $|v| = 1$ and let

$$
z := \langle Av, v \rangle \in \mathrm{Num}\,(A) \, .
$$

For all $t \geq 5$, let

$$
\psi_t(x) := \begin{cases} \frac{1 - \cos(\pi t x / 2)}{\sqrt{4 - 10/t}} & \text{if} \quad 0 \leq x \leq 2/t \\ \frac{2}{\sqrt{4 - 10/t}} & \text{if} \quad 2/t \leq x \leq 1 - 2/t \\ \frac{1 - \cos(\pi t (x-1)/2)}{\sqrt{4 - 10/t}} & \text{if} \quad 1 - 2/t \leq x \leq 1. \end{cases}
$$

Then $\psi_t(0) = \psi_t(1) = \psi_t'(0) = \psi_t'(1) = 0$,

$$
\int_0^1 |\psi_t(x)|^2 \mathrm{d}x = 1 \qquad \text{and} \qquad \int_0^1 |\psi_t'(x)|^2 \mathrm{d}x = \frac{\pi^2 t^2}{8t - 20}.
$$

Let $f_t := v\psi_t \in \mathrm{Dom}(D)$. By construction $\|f_t\| = 1$ and

$$
\begin{aligned}
\langle ADf_t, f_t \rangle &= \int_0^1 \langle Af_t'(x), f_t'(x) \rangle \, \mathrm{d}x \\
&= \langle Av, v \rangle \int_0^1 |\psi_t'(x)|^2 \mathrm{d}x \\
&= \frac{z\pi^2 t^2}{8t - 20}.
\end{aligned}
$$

Thus by taking $t \to \infty$, from the fact that $0 \in \mathrm{Num}\,(AD)$ and since $\mathrm{Num}\,(AD)$ is convex, we gather

$$
\overline{\mathrm{Num}\,(AD)} \supseteq \Phi.
$$

Case b): now $c \neq 0$. Let $z$ be a fixed non-zero complex number. Our aim is to find functions $f_\varepsilon \in \mathrm{Dom}(D)$ parameterized by $\varepsilon > 0$, such that $\|f_\varepsilon\| = 1$ and $\langle Af_\varepsilon, f_\varepsilon \rangle$ is close to $z$ for small $\varepsilon$.

For $0 < \varepsilon < 1/2$, let

$$
\phi_\varepsilon(x) := \begin{cases} \frac{\varepsilon}{c\pi} \sin(x\pi/\varepsilon) & \text{if } 0 \leq x \leq \varepsilon/2 \\ \frac{\varepsilon}{2c\pi}[1 - \cos(2\pi(x/\varepsilon - 1))] & \text{if } \varepsilon/2 \leq x \leq \varepsilon \\ 0 & \text{if } \varepsilon \leq x \leq 1. \end{cases}
$$

Then, straightforward computations show $\phi_\varepsilon(0) = \phi_\varepsilon(1) = \phi_\varepsilon'(1) = 0$, $\phi_\varepsilon'(0) = c^{-1}$,

$$
\int_0^1 |\phi_\varepsilon(x)|^2 \mathrm{d}x = \frac{11\varepsilon^3}{16c^2\pi^2} \qquad \text{and} \qquad \int_0^1 |\phi_\varepsilon'(x)|^2 \mathrm{d}x = \frac{\varepsilon}{2c^2}.
$$

For all $\varepsilon > 0$ small enough, we define the required test function $f_\varepsilon$ as

$$
f_\varepsilon(x) := \begin{pmatrix} z\phi_\varepsilon(x) \\ \alpha(\varepsilon) \end{pmatrix}
$$

where

$$
\alpha(\varepsilon) := \sqrt{1 - |z|^2 \|\phi_\varepsilon\|^2} = \sqrt{1 - \frac{11|z|^2\varepsilon^3}{16c^2\pi^2}}
$$

is independent of $x$. By construction $f_\varepsilon \in \mathrm{Dom}(D)$ and

$$
\|f_\varepsilon\|^2 = \|z\phi_\varepsilon\|^2 + \alpha(\varepsilon)^2 = 1.
$$

According to (3),

$$
\begin{aligned}
\langle ADf_\varepsilon, f_\varepsilon \rangle &= \left\langle A \begin{pmatrix} z\phi_\varepsilon'(0) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha(\varepsilon) \end{pmatrix} \right\rangle + \int_0^1 \langle Af'(x), f'(x) \rangle \, \mathrm{d}x \\
&= z\alpha(\varepsilon) + \left\langle A \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \int_0^1 |z|^2 |\phi_\varepsilon'(x)|^2 \mathrm{d}x \\
&= z\alpha(\varepsilon) + \frac{a\varepsilon}{2c^2}|z|^2.
\end{aligned}
$$

Since $\alpha(\varepsilon) \to 1$ as $\varepsilon \to 0$, the above shows $\langle ADf_\varepsilon, f_\varepsilon \rangle \to z$ as $\varepsilon \to 0$, so that $z$ is an accumulation point of $\mathrm{Num}\,(AD)$. By moving

$z \in \mathbb{C}$, any complex number is accumulation point of $\mathrm{Num}\,(AD)$. Since $\mathrm{Num}\,(AD)$ is convex, the only possibility for $\mathrm{Num}\,(AD)$ is to be the whole complex plane. ∎

## 3. The resolvent of $AD$

In this section we show that the resolvent of $AD$ is compact for all non-singular $A$. In general it is false that the product of a bounded operator and an operator whose resolvent is compact has compact resolvent, however if we know in addition that the spectrum of the product is not the whole of $\mathbb{C}$, then the assertion is true.

We first show that the resolvent of $D$ is compact by making use of its self-adjointness. Since the constant function

$$f_0 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is in $\mathrm{Dom}(D)$ and $AD f_0$ vanishes,

$$0 \in \mathrm{Spec}\,(AD)\,.$$

**Proposition 5.** *If $A$ is a diagonal matrix, then*

$$\mathrm{Spec}\,(AD) = \{a_-\pi^2 k^2, a_+\pi^2 k^2\}_{k=0}^{\infty}.$$

*The zero eigenvalue is always non-degenerate and all the remaining eigenvalues are of multiplicity no greater than 2.*

**Proof.** Let $f_0 \in \mathrm{Dom}(D)$ be as above. For all $n = 1, 2, \ldots$, let

$$f_{2n-1}(x) := \sqrt{2}\begin{pmatrix} \sin(\pi n x) \\ 0 \end{pmatrix} \quad \text{and} \quad f_{2n}(x) := \sqrt{2}\begin{pmatrix} 0 \\ \cos(\pi n x) \end{pmatrix}. \quad (4)$$

Then $f_k \in \mathrm{Dom}(D)$,

$$AD f_{2n-1} = (a_+\pi^2 n^2) f_{2n-1}, \qquad AD f_{2n} = (a_-\pi^2 n^2) f_{2n}$$

and $\{f_k\}_{k=0}^{\infty}$ is a complete orthonormal set in $\mathrm{L}^2(0,1) \otimes \mathbb{C}^2$. ∎

According to corollary 3 and the above proposition, $D = D^* \geq 0$ and

$$\mathrm{Spec}\,(D) = \{\pi^2 k^2\}_{k=0}^{\infty}.$$

Since the eigenfunctions $\{f_k\}_{k=0}^{\infty}$ form a complete orthonormal set, the resolvent of $D$ is compact.

Let us now rule out the possibility $\mathrm{Spec}\,(AD) = \mathbb{C}$.

**Lemma 6.** *For any non-singular $A \in \mathbb{C}$,*

$$\mathrm{Spec}\,(AD) \neq \mathbb{C}.$$

**Proof.** Fix the matrix $A$. Since
$$AD - \lambda = A(D - \lambda A^{-1}),$$
the complex number $\lambda \in \operatorname{Spec}(AD)$, if and only if
$$0 \in \operatorname{Spec}\left(D - \lambda A^{-1}\right).$$
Let $H(\lambda) := D - \lambda A^{-1}$. Then the family of operators $H(\lambda)$ with domain $\operatorname{Dom}(D)$ independent of $\lambda$ is a holomorphic family of type (A) for all $\lambda \in \mathbb{C}$. Since $0$ is a non-degenerate isolated eigenvalue of $H(0) = D$ and $A^{-1}$ is bounded, there exist an open neighbourhood $0 \in U \subset \mathbb{C}$ such that $H(\lambda)$ has a non-degenerate isolated eigenvalue, (denoted by $\mu(\lambda)$) close to $0$ for all $\lambda \in U$ and $\mu(\lambda)$ is a complex valued holomorphic function in $U$ (cf. [7, th.XII.8]).

If there exists some $\lambda_0 \in U$ satisfying $\mu(\lambda_0) \neq 0$, then $0 \notin \operatorname{Spec}(H(\lambda_0))$ so that $\lambda_0 \notin \operatorname{Spec}(AD)$. Hence, in order to show that $\operatorname{Spec}(AD) \neq \mathbb{C}$, it is enough to show that $\mu \not\equiv 0$. For this we find the first coefficients in the Rayleigh-Schrödinger series expansion of $\mu$ about $0$. Let
$$\mu(\lambda) = \mu_0 + \mu_1 \lambda + \mu_2 \lambda^2 + \ldots \qquad\qquad \lambda \in U.$$
Since $\mu(0) = 0$, $\mu_0 = 0$. Since $\|f_0\| = 1$ and $H(0)f_0 = Df_0 = 0$, we compute directly $\mu_1$ (cf. [6, remark 2.2, p.80]) by
$$\begin{aligned}
\mu_1 &= \langle A^{-1}f_0, f_0 \rangle \\
&= \left\langle A^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.
\end{aligned}$$
If $A$ is such that $a \neq 0$,
$$\left\langle A^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \neq 0$$
so that $\mu_1$ does not vanish and hence $\mu \not\equiv 0$.

Let $A$ be such that $a = 0$. Then $\mu_1 = 0$ so we compute $\mu_2$. Let $f_k$ be the eigenfunctions of $D$ as in (4) so that $\|f_k\| = 1$ for all $k = 1, 2, \ldots$. Let $\lambda_{2n-1} = \lambda_{2n} := \pi^2 n^2$ for all $n = 1, 2, \ldots$ so that
$$H(0)f_k = Df_k = \lambda_k f_k.$$
Then (cf. [6, remark 2.2, p.80])
$$-\mu_2 = \sum_{k=1}^{\infty} \frac{\langle A^{-1}f_0, f_k \rangle \langle A^{-1}f_k, f_0 \rangle}{\lambda_k}.$$
We compute each term in the series. Since $a = 0$ and $A$ is invertible, then $b$ and $c$ do not vanish and
$$A^{-1} = \begin{pmatrix} -d/(bc) & 1/c \\ 1/b & 0 \end{pmatrix}.$$
Hence
$$\langle A^{-1}f_0, f_k \rangle = \int_0^1 \left\langle A^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, f_k(x) \right\rangle \mathrm{d}x = \int_0^1 \left\langle \begin{pmatrix} 1/c \\ 0 \end{pmatrix}, f_k(x) \right\rangle \mathrm{d}x,$$

so that

$$\left\langle A^{-1}f_0, f_{2n}\right\rangle = \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} 1/c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \cos(\pi nx) \end{pmatrix} \right\rangle \mathrm{d}x = 0$$

and

$$\begin{aligned}
\left\langle A^{-1}f_0, f_{2n-1}\right\rangle &= \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} 1/c \\ 0 \end{pmatrix}, \begin{pmatrix} \sin(\pi nx) \\ 0 \end{pmatrix} \right\rangle \mathrm{d}x \\
&= \sqrt{2}/c \int_0^1 \sin(\pi nx)\mathrm{d}x \\
&= \begin{cases} 0 & \text{if } n = 2m \\ 2\sqrt{2}/(c\pi n) & \text{if } n = 2m - 1 \end{cases}
\end{aligned}$$

for $m$ integer and $n = 1, 2, \ldots$. On the other hand

$$\left\langle A^{-1}f_k, f_0\right\rangle = \int_0^1 \left\langle A^{-1}f_k(x), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \mathrm{d}x,$$

so that

$$\begin{aligned}
\left\langle A^{-1}f_{2n}, f_0\right\rangle &= \sqrt{2}\int_0^1 \left\langle A^{-1}\begin{pmatrix} 0 \\ \cos(\pi nx) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \mathrm{d}x \\
&= \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} \cos(\pi nx)/c \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \mathrm{d}x = 0
\end{aligned}$$

and

$$\begin{aligned}
\left\langle A^{-1}f_{2n-1}, f_0\right\rangle &= \sqrt{2}\int_0^1 \left\langle A^{-1}\begin{pmatrix} \sin(\pi nx) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \mathrm{d}x \\
&= \sqrt{2}\int_0^1 \left\langle \begin{pmatrix} d\sin(\pi nx)/(bc) \\ \sin(\pi nx)/b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \mathrm{d}x \\
&= \sqrt{2}/(b)\int_0^1 \sin(\pi nx)\mathrm{d}x \\
&= \begin{cases} 0 & \text{if } n = 2m \\ 2\sqrt{2}/(b\pi n) & \text{if } n = 2m - 1 \end{cases}
\end{aligned}$$

for $m$ integer and $n = 1, 2, \ldots$. This yields

$$\left\langle A^{-1}f_0, f_k\right\rangle\left\langle A^{-1}f_k, f_0\right\rangle = \begin{cases} 0 & \text{if } k \neq 4m - 3 \\ 8/(bc\pi^2 n^2) & \text{if } k = 4m - 3 \end{cases}$$

for $m = 1, 2, \ldots$. Thus

$$-\mu_2 = \sum_{m=1}^{\infty} \frac{8}{bc\pi^4(2m-1)^4} \neq 0$$

so that $\mu \not\equiv 0$ as we required. ∎

**Theorem 7.** *For all $z \notin \mathrm{Spec}\,(AD)$, the resolvent $(AD - z)^{-1}$ is compact.*

**Proof.** Since $D$ is non-negative and it has compact resolvent,

$$AD + A = A(D + 1)$$

has a compact inverse. Let $z \notin \operatorname{Spec}(AD)$, then

$$
\begin{aligned}
AD - z &= AD + A - A - z \\
&= \left(I - (A + z)(AD + A)^{-1}\right)(AD + A).
\end{aligned}
$$

Hence

$$(AD + A)^{-1} = (AD - z)^{-1}(I - (A + z)(AD + A)^{-1}),$$

so that

$$
\begin{aligned}
(AD - z)^{-1} &= (AD - z)^{-1}(A + z)(AD + A)^{-1} + (AD + A)^{-1} \\
&= ((AD - z)^{-1}(A + z) + 1)(AD + A)^{-1}.
\end{aligned}
$$

Thus $(AD - z)^{-1}$ is compact as needed. ∎

Theorem 7 shows that the spectrum of $AD$ consists entirely of isolated eigenvalues of finite multiplicity. Since the eigenvalue problem $ADf = \lambda f$ is a constant coefficient system of second order ordinary differential equations, due to the fact that we have a combination Dirichlet and Neumann boundary condition at both ends of the interval, the multiplicity of each eigenvalue is never greater than 2.

## 4. Asymptotics of the resolvent

We now investigate the asymptotic behaviour of the resolvent norm of $AD$. The results we discuss in this section are connected with the stability of the heat semigroup $\mathrm{e}^{-ADt}$. They are also relevant from the computational point of view and they are closely related to both local and global stability of the spectrum (cf. [1], [4], [9] and the reference therein). The present approach is motivated by analogous reports on non-self-adjoint Schrödinger operators in [2], [4] and [5].

Let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Below and elsewhere we will denote by $\tilde{D} := JD$. According to lemma 3, $\tilde{D} = \tilde{D}^*$. According to lemma 5,

$$\operatorname{Spec}\left(\tilde{D}\right) = \{\pm \pi^2 n^2\}_{n=0}^{\infty}$$

each eigenvalue being of multiplicity 1. We will employ part b) of the following theorem in the proof of theorem 21-b).

**Theorem 8.** *Assume that there exists a non-singular diagonal matrix $B$ such that $B^{-1}AB = (B^{-1}AB)^* > 0$. Then*

  a) *$AD$ is similar to a non-negative self-adjoint operator.*

b) $A\tilde{D}$ *is similar to a self-adjoint operator whose numerical range is the whole real line.*

**Proof.** Let $C := B^{-1}AB$ so that $C = C^* > 0$. Since diagonal matrices commute with the boundary conditions, $AD$ is similar to $CD$. For the same reason and since diagonal matrices also commute with $J$, $A\tilde{D}$ is similar to $C\tilde{D}$.

By hypothesis, the square root $C^{1/2} = (C^{1/2})^* > 0$. Then

$$CD = C^{1/2}(C^{1/2}DC^{1/2})C^{-1/2} = C^{1/2}KC^{-1/2},$$

where

$$K = C^{1/2}DC^{1/2}$$

$$\text{Dom}(K) = \{f \in L^2(0,1) \otimes \mathbb{C}^2 : C^{1/2}f \in \text{Dom}(D)\},$$

so that $CD$ is similar to $K$. Since $D = D^* \geq 0$, then $K = K^* \geq 0$.

Analogously $C\tilde{D}$ is similar to

$$\tilde{K} := C^{1/2}\tilde{D}C^{1/2}$$

where $\text{Dom}(\tilde{K}) = \text{Dom}(K)$. Since $\tilde{D} = \tilde{D}^*$, then $\tilde{K} = \tilde{K}^*$. Furthermore, since

$$\text{Num}\left(\tilde{D}\right) = \mathbb{R}$$

and

$$\left\langle \tilde{K}f, f \right\rangle = \left\langle \tilde{D}C^{1/2}f, C^{1/2}f \right\rangle,$$

the numerical range of $\tilde{K}$ is the whole real line. ∎

Let $A$ be as in the hypothesis. The similarity to a self-adjoint operator ensures the existence of a constant $k_A \geq 1$ such that

$$\|(AD - z)^{-1}\| \leq \frac{k_A}{\text{dist}\left(z, [0, \infty)\right)} \qquad z \notin \text{Spec}(AD)$$

and

$$\|(A\tilde{D} - z)^{-1}\| \leq \frac{k_A}{\text{dist}\left(z, \mathbb{R}\right)} \qquad z \notin \text{Spec}\left(A\tilde{D}\right).$$

These identities show that although the numerical range of $AD$ and $A\tilde{D}$ are in general the whole complex plane, the eigenvalues of these operators are stable in the sense of [9].

If we assume the weaker condition $C + C^* > 0$, we show how to recover part of the above estimate. We start with a preliminary lemma.

**Lemma 9.** *Let $A$ be such that* $\text{Num}(A) \subset \{\text{Re}(z) > 0\}$. *Then* $\text{Spec}(AD) \subset \{\text{Re}(z) \geq 0\}$ *and there exists $k > 0$ independent of $z$, such that*

$$\|(AD - z)^{-1}\| \leq \frac{k}{|z|} \qquad\qquad \text{Re}(z) < 0. \qquad\qquad (5)$$

**Proof.** Let $r > 0$ and let $z \notin [0, \infty)$. Then

$$
\begin{aligned}
AD - z &= A(D - zA^{-1}) \\
&= A\left[(D - rz) + z(r - A^{-1})\right] \\
&= A\left[1 + (r - A^{-1})z(D - rz)^{-1}\right](D - rz).
\end{aligned}
$$

Therefore $z \notin \operatorname{Spec}(AD)$, whenever

$$
\left\|(r - A^{-1})z(D - rz)^{-1}\right\| < 1. \tag{6}
$$

We show that there is always $r > 0$ independent of $z$, such that this holds for all $\operatorname{Re}(z) < 0$.

Since $D \geq 0$ and $0 \in \operatorname{Spec}(AD)$,

$$
\|(D - rz)^{-1}\| = \frac{1}{r|z|}.
$$

Thus

$$
\left\|(r - A^{-1})z(D - rz)^{-1}\right\| \leq \|1 - r^{-1}A^{-1}\|.
$$

The hypothesis we imposed on $A$ is equivalent to saying

$$
A + A^* > 0,
$$

then

$$
A^{-1} + (A^{-1})^* = A^{-1}(A^* + A)(A^{-1})^* > 0.
$$

For all $v \in \mathbb{C}^2$,

$$
\begin{aligned}
\|(I - r^{-1}A^{-1})v\|^2 &= \left\langle \left(I - r^{-1}(A^{-1} + (A^{-1})^*) + r^{-2}(A^{-1})^*A^{-1}\right)v, v\right\rangle \\
&= |v|^2 - r^{-1}\left\langle \left(A^{-1} + (A^{-1})^* + r^{-1}(A^{-1})^*A^{-1}\right)v, v\right\rangle.
\end{aligned}
$$

Hence there exists a constant $k_0 > 0$ independent of $r$ (and $z$), such that

$$
\|I - r^{-1}A^{-1}\| < 1 - r^{-1}k_0
$$

when $r$ is large enough. For such an $r$, identity (6) holds for any $\operatorname{Re}(z) < 0$. This shows that $\operatorname{Spec}(AD)$ must be enclosed in the right hand plane. Furthermore

$$
\begin{aligned}
\|(AD - z)^{-1}\| &\leq \|A^{-1}\|\|(D - zA^{-1})^{-1}\| \\
&\leq \|A^{-1}\|\|(D - rz)^{-1}\|\left\|\left(1 + z(D - rz)^{-1}(r - A^{-1})\right)^{-1}\right\| \\
&\leq \frac{\|A^{-1}\|}{r|z|}\sum_{l=0}^{\infty}\|z(D - rz)^{-1}(r - A^{-1})\|^l \\
&\leq \frac{\|A^{-1}\|}{r|z|}\sum_{l=0}^{\infty}\|1 - r^{-1}A^{-1}\|^l \\
&\leq \frac{k}{|z|}
\end{aligned}
$$

so (5) is also proven. ■

Below and elsewhere we denote by $\Omega$ the set of non-singular diagonal matrices and

$$S(\alpha, \beta) := \{z \in \mathbb{C} : \alpha \leq \arg(z) \leq \beta\} \qquad \alpha \leq \beta.$$

**Theorem 10.** *If there exists $B \in \Omega$ such that*

$$\operatorname{Num}\left(B^{-1}AB\right) \subset S(\alpha, \beta) \qquad \beta - \alpha < \pi,$$

*then $\operatorname{Spec}(AD) \subset S(\alpha, \beta)$ and for any small enough $\varepsilon > 0$ there exists $k_\varepsilon > 0$ independent of $z$, such that*

$$\|(AD - z)^{-1}\| \leq \frac{k_\varepsilon}{|z|} \qquad z \notin S(\alpha - \varepsilon, \beta + \varepsilon).$$

**<u>Proof.</u>** Let $C := B^{-1}AB$, so that

$$\operatorname{Num}(C) \subset S(\alpha, \beta).$$

Since $B$ commutes with the boundary conditions, $AD$ is similar to $CD$ and so it is enough to show the theorem for $CD$. Now, for all $-(\alpha + \pi/2) < \vartheta < \pi/2 - \beta$

$$\operatorname{Num}\left(e^{i\vartheta}C\right) \subset \{\operatorname{Re}(z) > 0\},$$

so we just have to apply lemma 9 to $e^{i\vartheta}C$. ∎

The constant $k_\varepsilon$ of this theorem is in general strictly greater than 1, therefore this is weaker than the similar condition for m-sectorial operators in [6, p.279].

If $A$ is triangular, the hypothesis of the above theorem does not necessarily hold. For instance if

$$A = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix} \qquad a > 0,$$

then

$$\operatorname{Num}(A) = \{a + z : |z| < 1/2\}$$

and so for small $a$ the numerical range contains the origin. Nonetheless by using a similarity transformation and an approximation argument, we can show positivity of the spectrum whenever both of the eigenvalues of $A$ are positive ($a > 0$ in our example). The conclusion about the spectrum of the following result will be improved in theorem 20.

**Corollary 11.** *Let $A$ be either upper or lower triangular. If $a \geq d > 0$, then*

$$\operatorname{Spec}(AD) \subset [0, \infty)$$

*and for all $\varepsilon > 0$ there exists $k_\varepsilon > 0$ independent of $z$, such that*

$$\|(AD - z)^{-1}\| < \frac{k_\varepsilon}{|z|}$$

*for all $z \notin S(-\varepsilon, \varepsilon)$.*

**Proof.** If $A$ is upper triangular the proof is similar so let us assume that

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

Let

$$A(r) := \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ rc & d \end{pmatrix}.$$

Then $AD$ is similar to $A(r)D$ for all $r \neq 0$. Put

$$C(r) := A(r) + A(r)^* = \begin{pmatrix} 2a & r\bar{c} \\ rc & 2d \end{pmatrix}.$$

Then $C(r) = C(r)^*$. The eigenvalues of $C(r)$ are

$$a + d \pm \sqrt{(a-d)^2 + r^2|c|^2},$$

thus for small $r > 0$, $C(r) > 0$. The numerical range of $A(r)$ is an ellipse with focus at $a, d$ and principal axis in the vertical direction of the order of $r$. By taking $r \to 0$, theorem 10 completes the proof.  ∎

If $A$ is as in the hypothesis of corollary 11, there does not exist $B \in \Omega$ such that $B^{-1}AB = (B^{-1}AB)^*$ or $B^{-1}(AJ)B = (B^{-1}(AJ)B)^*$ so theorem 8 is not applicable. We show that at least in one case $AD$ fails to be similar to self-adjoint.

**Theorem 12.** *Let*

$$A = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix} \qquad a > 0.$$

*Let $\varepsilon > 0$ and $z(r) := 4a\pi^2 r^2 \pm i\varepsilon$. Then there exists a constant $k_\varepsilon > 0$ independent of $r$, such that*

$$\|(AD - z(r))^{-1}\| > k_\varepsilon r^{1/2} \qquad r = 1, 2, \ldots.$$

**Proof.** Fix $\varepsilon > 0$ and let $z(r) := 4a\pi^2 r^2 - i\varepsilon$. Without loss of generality we can assume $r = 3, 4, \ldots$. Throughout the proof the constants $l_j$ are assumed to be positive, possibly depending upon $\varepsilon$ but independent of $r$. In order to show the desired conclusion, it is enough to find $f_r \in \mathrm{Dom}(D)$ and $l_0$, such that

$$\frac{\|ADf_r - z(r)f_r\|}{\|f_r\|} \leq l_0 r^{-1/2} \qquad (7)$$

for all large enough $r$.

Let

$$f = \begin{pmatrix} \phi \\ \gamma \end{pmatrix} \in \mathrm{Dom}(D).$$

Then

$$ADf - z(r)f = -\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}\begin{pmatrix} \phi'' \\ \gamma'' \end{pmatrix} - z(r)\begin{pmatrix} \phi \\ \gamma \end{pmatrix}$$

$$= \begin{pmatrix} -a\phi'' - z(r)\phi \\ -\phi'' - a\gamma'' - z(r)\gamma \end{pmatrix}.$$

Hence

$$\|f\|^2 = \|\phi\|^2 + \|\gamma\|^2$$

and

$$\|ADf - z(r)f\|^2 = \|a\phi'' + z(r)\phi\|^2 + \|a\gamma'' + z(r)\gamma + \phi''\|^2.$$

We now define the appropriate $f_r \in \mathrm{Dom}(D)$ satisfying (7). Let

$$\gamma_r(x) := \cos(2\pi rx)$$

Then $\|\gamma_r\|^2 = 1/2$. Let

$$\phi_r(x) := \begin{cases} -i\varepsilon\cos(2\pi rx)/(4\pi^2 r^2) & \text{if} \quad x \in (1/r, 1 - 1/r) \\ 0 & \text{if} \quad x \notin (1/(2r), 1 - 1/(2r)) \end{cases}$$

be such that $\phi_r$ is smooth and

   a) $|\phi_r(x)| \le \varepsilon/(4\pi^2 r^2)$ for all $x \in [0,1]$,
   b) $|\phi_r'(x)| \le l_1/r$ for all $x \notin (1/r, 1 - 1/r)$,
   c) $|\phi_r''(x)| \le l_2$ for all $x \notin (1/r, 1 - 1/r)$.

Then

$$\|\phi_r\|^2 = \int_0^1 |\phi_r(x)|^2 \mathrm{d}x \le \varepsilon^2/(16\pi^4 r^4) \le l_3 r^{-4}$$

and

$$\|\phi_r\|^2 \ge \int_{1/r}^{1-1/r} \frac{\varepsilon^2 \cos^2(2\pi rx)}{16\pi^4 r^4}\mathrm{d}x \ge l_4 r^{-4}.$$

Hence

$$f_r = \begin{pmatrix} \phi_r \\ \gamma_r \end{pmatrix} \in \mathrm{Dom}(D)$$

and

$$1/2 \le \|f_r\|^2 = \|\phi_r\|^2 + \|\gamma_r\|^2 \le 1 \tag{8}$$

for all large enough $r$. If $1/r < x < 1 - 1/r$,

$$a\phi''(x) + z(r)\phi(x) = a\phi(x)'' + 4a\pi^2 r^2\phi(x) - i\varepsilon\phi(x)$$

$$= [ai\varepsilon\cos(2\pi rx)'' + 4a\pi^2 r^2 i\varepsilon\cos(2\pi rx) +$$

$$- \varepsilon^2\cos(2\pi rx)]/(4\pi^2 r^2)$$

$$= -\varepsilon^2\cos(2\pi rx)/(4\pi^2 r^2).$$

Then, a) and c) yield

$$\|a\phi'' + z(r)\phi\|^2 = \int_0^1 |a\phi''(x) + z(r)\phi(x)|^2 \mathrm{d}x$$

$$\leq \int_{1/r}^{1-1/r} l_5/r^4 \mathrm{d}x + \int_{x \notin [1/r, 1-1/r]} l_6 + l_7/r^4 \mathrm{d}x$$

$$\leq l_6 r^{-1} + l_5 r^{-4} + l_7 r^{-5}. \tag{9}$$

Also,

$$a\gamma''(x) + z(r)\gamma(x) + \phi''(x) = a\gamma''(x) + 4a\pi^2 r^2 \gamma(x) - i\varepsilon\gamma(x) + \phi''(x)$$

$$= a\cos(2\pi rx)'' + 4a\pi^2 r^2 \cos(2\pi rx) +$$

$$- i\varepsilon\cos(2\pi rx) + \phi''(x)$$

$$= \phi''(x) - i\varepsilon\cos(2\pi rx).$$

Then for $1/r < x < 1 - 1/r$,

$$a\gamma''(x) + z(r)\gamma(x) + \phi''(x) = i\varepsilon\cos(2\pi rx) - i\varepsilon\cos(2\pi rx) = 0$$

and thus c) yields

$$\|a\gamma'' + z(r)\gamma + \phi''\|^2 = \int_0^1 |a\gamma''(x) + z(r)\gamma(x) + \phi''(x)|^2 \mathrm{d}x$$

$$= \int_{x \notin [1/r, 1-1/r]} |i\varepsilon\cos(2\pi rx) - \phi''(x)|^2 \mathrm{d}x$$

$$\leq \int_{x \notin [1/r, 1-1/r]} l_8 \mathrm{d}x$$

$$\leq l_8 r^{-1}. \tag{10}$$

In order to complete the proof for $z(r) := 4a\pi r^2 - i\varepsilon$, notice that (8), (9) and (10), show (7). On the other hand, if $z(r) := 4a\pi r^2 + i\varepsilon$ it is enough to substitute $\phi_r$ by $-\phi_r$ and repeat the above computations. ■

This result is still valid for

$$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Indeed, it is enough to put $\phi_r(x) := \sin(2\pi rx)$,

$$\gamma_r(x) := \begin{cases} \pm i\varepsilon \sin(2\pi rx)/(4\pi^2 r^2) & \text{if} \quad x \in (1/r, 1 - 1/r) \\ 0 & \text{if} \quad x \notin (1/(2r), 1 - 1/(2r)) \end{cases}$$

and carry out similar calculations. Since the resolvent norm of self-adjoint operators remains bounded in horizontal lines, the above $AD$ can not be similar to any self-adjoint operator.

Let $\Omega_r$ be the set of all non-degenerate real diagonal matrices. If $A$ does not satisfy the hypothesis of theorem 10 (for instance the numerical range of $A$ is an ellipse centered at the origin), but $A$ is "close" in

some sense to $\Omega_r$, an alternative to theorem 10 can be established. We will employ this result in the proof of theorem 20.

**Theorem 13.** *Let there exist $B \in \Omega_r$ such that*

$$\|AB - I\| < 1.$$

*Let $\omega := \arcsin(\|AB - I\|)$ with $0 \leq \omega < \pi/2$. Then*

$$\mathrm{Spec}(AD) \subset S(-\omega, \omega) \cup S(-\pi - \omega, \omega - \pi)$$

*and for any small enough $\varepsilon > 0$ there exist $k_\varepsilon > 0$ independent of $z$, such that*

$$\|(AD - z)^{-1}\| \leq \frac{k_\varepsilon}{|z|}$$

*for all $z \notin S(-\omega - \varepsilon, \omega + \varepsilon) \cup S(-\pi - \omega - \varepsilon, \omega - \pi + \varepsilon)$.*

**Proof.** If $\omega = 0$, $A \in \Omega_r$ so the conclusion is consequence of corollary 3. Let $\omega > 0$, let $l := \|AB - I\|$ and put $C := B^{-1} \in \Omega_r$. Then $CD = (CD)^*$ and according to the hypothesis $0 < l < 1$.

Let $z \in \mathbb{C}$ be such that $z \notin S(-\omega, \omega) \cup S(-\pi - \omega, \omega - \pi)$. Then

$$\begin{aligned}
(AD - z) &= CD + (A - C)D - z \\
&= [I + (AB - I)CD(CD - z)^{-1}](CD - z).
\end{aligned}$$

Since $CD$ is self-adjoint and by definition $w = \arcsin(l)$,

$$\begin{aligned}
\|(AB - I)CD(CD - z)^{-1}\| &\leq l\|CD(CD - z)^{-1}\| \\
&\leq l \sup_{x \in \mathbb{R}} \left| \frac{x}{x - z} \right| \\
&\leq \sup_{x \in \mathbb{R}} \frac{l}{\left| 1 - \frac{z}{x} \right|} < 1, \quad (11)
\end{aligned}$$

so that

$$[I + (AB - I)CD(CD - z)^{-1}]$$

is invertible. Hence

$$z \notin \mathrm{Spec}(AD)$$

and

$$(AD - z)^{-1} = (CD - z)^{-1}[I + (AB - I)CD(CD - z)^{-1}]^{-1} \quad (12)$$

for all $\omega < |\arg(z)| \leq \pi$. This encloses $\mathrm{Spec}(AD)$.

In order to show the second part, let

$$z \notin S(-\omega - \varepsilon, \omega + \varepsilon) \cup S(-\pi - \omega - \varepsilon, \omega - \pi + \varepsilon),$$

for small $\varepsilon > 0$. Then there exist a constant $l_1(\varepsilon) > 0$ independent of $z$, such that

$$\|(CD - z)^{-1}\| \leq \frac{l_1(\varepsilon)}{|z|}.$$

Also, there exist a constant $0 < l_2(\varepsilon) < 1$ independent of $z$, such that

$$\sup_{x \in \mathbb{R}} \frac{l}{\left|1 - \frac{z}{x}\right|} < l_2(\varepsilon).$$

These two estimates, (11) and (12) yield

$$\|(AD - z)^{-1}\| \leq \frac{l_1(\varepsilon) \sum_{n=0}^{\infty} l_2(\varepsilon)^n}{|z|} = \frac{k_\varepsilon}{|z|}. \quad \blacksquare$$

This shows that if $A_n \in \mathbb{C}^{2 \times 2}$ is a sequence of non-singular matrices and there exists $B \in \Omega_r$ such that

$$\|A_n - B\| \to 0$$

as $n \to \infty$, then

$$\mathrm{Spec}(A_n D) \to \mathbb{R}.$$

**Corollary 14.** *Let $A$ be either upper or lower triangular. If $a, d \in \mathbb{R}$ and $ad < 0$, then*

$$\mathrm{Spec}(AD) \subset \mathbb{R}$$

*and for all $\varepsilon > 0$ there exists $k_\varepsilon > 0$, such that*

$$\|(AD - z)^{-1}\| < \frac{k_\varepsilon}{|z|}$$

*for all $z \notin S(-\varepsilon, \varepsilon) \cup S(-\pi - \varepsilon, \varepsilon - \pi)$.*

**Proof.** It is similar to the proof of corollary 11. Assume without loss of generality that $b = 0$ and let

$$A(r) = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ rc & d \end{pmatrix}.$$

Then $AD$ is similar to $A(r)D$ for all $r > 0$. Put

$$C = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \in \Omega_r,$$

then

$$\|A(r)C - I\| = \left\| \begin{pmatrix} 0 & 0 \\ rc/a & 0 \end{pmatrix} \right\| = r|c/a|.$$

Let $\omega_r := \arcsin(r|c/a|)$. According to theorem 13, for all $0 < r < |a/c|$

$$\begin{aligned} \mathrm{Spec}(AD) &= \mathrm{Spec}(A(r)D) \\ &\subset S(-\omega_r, \omega_r) \cup S(-\pi - \omega_r, \omega_r - \pi). \end{aligned}$$

By taking $r$ small enough, theorem 13 yields the desired estimate for the resolvent norm. By taking $r \to 0$, *a fortiori* $\mathrm{Spec}(AD) \subset \mathbb{R}$. $\quad \blacksquare$

## 5. The Hamiltonian ODE system

In this section we find an entire function whose zeros coincide with $\mathrm{Spec}(AD)$. This is made by computing the transcendental function associated with the $2 \times 2$ system of ordinary differential equations associated to $AD$ via standard ODE arguments.

Let the $2 \times 2$ constant coefficients second order eigenvalue problem

$$-Af'' = \lambda^2 f \tag{13}$$

$$\begin{aligned}
Pf(0) + (I - P)f'(0) &= 0 \\
Pf(1) + (I - P)f'(1) &= 0.
\end{aligned} \tag{14}$$

We will say that the complex number $\lambda$ is an eigenvalue of the system (13)-(14), if there exist a non-vanishing $f \in C^\infty(0,1) \otimes \mathbb{C}^2$ satisfying (13) and the boundary conditions (14). By regularity, $\lambda^2$ is an eigenvalue of $AD$, if and only if $\lambda$ is an eigenvalue of (13)-(14). Our aim is to find a holomorphic function, denoted by $EV(x)$ below, whose zeros coincide with the eigenvalues of (13)-(14).

We proceed in the classical manner. Let the decomposition in Jordan canonical form of $A$ be

$$A =: VCV^{-1}$$

where the Jordan matrix $C$ is either

$$C = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix} \qquad \text{or} \qquad C = \begin{pmatrix} a_+ & 0 \\ 1 & a_+ \end{pmatrix}$$

and

$$V := \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}.$$

Then (13)-(14) is equivalent to the $2 \times 2$ system

$$-Cg'' = \lambda^2 g \tag{15}$$

$$\begin{aligned}
PVg(0) + (I - P)Vg'(0) &= 0 \\
PVg(1) + (I - P)Vg'(1) &= 0.
\end{aligned} \tag{16}$$

In order to solve (15)-(16), we reduce it to a first order $4 \times 4$ system as follows. For all $\lambda \in \mathbb{C}$, let

$$B_\lambda = \begin{pmatrix} 0 & I \\ -\lambda^2 C^{-1} & 0 \end{pmatrix} \in \mathbb{C}^{4 \times 4}$$

and let

$$\Psi := \begin{pmatrix} v_1 & v_2 & 0 & 0 \\ 0 & 0 & v_3 & v_4 \end{pmatrix} \in \mathbb{C}^{2 \times 4}.$$

By regarding

$$\Phi = \begin{pmatrix} g \\ g' \end{pmatrix} \in \mathbb{C}^4,$$

one sees that (15)-(16) is equivalent to

$$\Phi' = B_\lambda \Phi \tag{17}$$

$$\Psi\Phi(0) = \Psi\Phi(1) = 0. \tag{18}$$

In order to solve (17)-(18) we must find a fundamental system of solutions. Let $e_1, e_2, e_3, e_4$ be the standard orthonormal basis of the Euclidean space $\mathbb{C}^4$. A straightforward computation show that

$$\exp(B_\lambda x)e_j \qquad x \in [0,1], \quad j = 1, 2, 3, 4$$

is indeed a linearly independent fundamental system for (17)-(18). Hence, $\lambda$ is an eigenvalue of this system, if and only if there exist $k_1, k_2, k_3, k_4$, such that

$$\Phi(x) = \sum_{j=1}^4 k_j \exp(B_\lambda x)e_j \tag{19}$$

is non-vanishing and satisfies the boundary conditions.

We now proceed to compute $EV(x)$. The exponential of $B_\lambda x$ is given by

$$\exp(B_\lambda x) = \begin{pmatrix} \cos(\lambda C^{-1/2}x) & \lambda^{-1}C^{1/2}\sin(\lambda C^{-1/2}x) \\ -\lambda C^{-1/2}\sin(\lambda C^{-1/2}x) & \cos(\lambda C^{-1/2}x) \end{pmatrix}$$

for $x \in [0,1]$. In theorems 15 and 16 below, we split our computation into two cases depending upon the Jordan matrix $C$.

**Theorem 15.** *When*

$$C = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix},$$

*$\lambda$ is an eigenvalue of the system (17)-(18), if and only if $EV(\lambda) = 0$ for*

$$EV(x) := \left(2\prod_{j=1}^4 v_j\right)\left[1 - \cos\left(\frac{x}{\sqrt{a_+}}\right)\cos\left(\frac{x}{\sqrt{a_-}}\right)\right] +$$

$$- \left[v_1^2 v_4^2 \frac{\sqrt{a_+}}{\sqrt{a_-}} + v_2^2 v_3^2 \frac{\sqrt{a_-}}{\sqrt{a_+}}\right]\sin\left(\frac{x}{\sqrt{a_+}}\right)\sin\left(\frac{x}{\sqrt{a_-}}\right).$$

**Proof.** Notice that $EV(0) = 0$. Assume $\lambda \neq 0$. According to the hypothesis,

$$C^{1/2} = \begin{pmatrix} a_+^{1/2} & 0 \\ 0 & a_-^{1/2} \end{pmatrix} \qquad \text{and} \qquad C^{-1/2} = \begin{pmatrix} a_+^{-1/2} & 0 \\ 0 & a_-^{-1/2} \end{pmatrix}.$$

Then

$$
\exp(B_\lambda x) = \begin{pmatrix}
\cos\frac{\lambda x}{\sqrt{a_+}} & 0 & \frac{\sqrt{a_+}}{\lambda}\sin\frac{\lambda x}{\sqrt{a_+}} & 0 \\
0 & \cos\frac{\lambda x}{\sqrt{a_-}} & 0 & \frac{\sqrt{a_-}}{\lambda}\sin\frac{\lambda x}{\sqrt{a_-}} \\
-\frac{\lambda}{\sqrt{a_+}}\sin\frac{\lambda x}{\sqrt{a_+}} & 0 & \cos\frac{\lambda x}{\sqrt{a_+}} & 0 \\
0 & -\frac{\lambda x}{\sqrt{a_-}}\sin\frac{\lambda x}{\sqrt{a_-}} & 0 & \cos\frac{\lambda x}{\sqrt{a_-}}
\end{pmatrix}.
$$

Let $\Phi(x)$ be a particular solution given as in (19), where the complex parameters $k_j$ are to be determined. Then

$$
\Psi\Phi(0) = \begin{pmatrix} k_1 v_1 + k_2 v_2 \\ k_3 v_3 + k_4 v_4 \end{pmatrix}
$$

and

$$
\Psi\Phi(1) = \begin{pmatrix}
k_1 v_1 \cos\frac{\lambda}{\sqrt{a_+}} + k_2 v_2 \cos\frac{\lambda}{\sqrt{a_-}} + \\
\qquad +k_3 v_1 \frac{\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}} + k_4 v_2 \frac{\sqrt{a_-}}{\lambda}\sin\frac{\lambda}{\sqrt{a_-}} \\
-k_1 v_3 \frac{\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} - k_2 v_4 \frac{\lambda}{\sqrt{a_-}}\sin\frac{\lambda}{\sqrt{a_-}} + \\
\qquad +k_3 v_3 \cos\frac{\lambda}{\sqrt{a_+}} + k_4 v_4 \cos\frac{\lambda}{\sqrt{a_-}}
\end{pmatrix}.
$$

The solution $\Phi$ satisfies the boundary conditions (18), if and only if

$$
\begin{cases}
k_1 v_1 + k_2 v_2 = 0 \\
k_3 v_3 + k_4 v_4 = 0 \\
k_1 v_1 \cos\frac{\lambda}{\sqrt{a_+}} + k_2 v_2 \cos\frac{\lambda}{\sqrt{a_-}} + k_3 \frac{v_1\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}} + k_4 \frac{v_2\sqrt{a_-}}{\lambda}\sin\frac{\lambda}{\sqrt{a_-}} = 0 \\
-k_1 \frac{v_3\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} - k_2 \frac{v_4\lambda}{\sqrt{a_-}}\sin\frac{\lambda}{\sqrt{a_-}} + k_3 v_3 \cos\frac{\lambda}{\sqrt{a_+}} + k_4 v_4 \cos\frac{\lambda}{\sqrt{a_-}} = 0.
\end{cases}
$$

The determinant of this $4\times4$ system of linear equations in $k_j$ is precisely $EV(\lambda)$. ∎

**Theorem 16.** *When*

$$
C = \begin{pmatrix} a_+ & 0 \\ 1 & a_+ \end{pmatrix},
$$

*$\lambda$ is an eigenvalue of the system (17)-(18), if and only if $EV(\lambda) = 0$ for*

$$
EV(x) := \left(\frac{v_2^2 v_4^2}{4a_+^3}\right) x^2 - \left(\det V + \frac{v_2 v_4}{2a_+}\right)^2 \sin^2\frac{x}{\sqrt{a_+}}.
$$

**<u>Proof.</u>** Notice that $EV(0) = 0$. Assume $\lambda \neq 0$. One can verify directly that

$$
C^{1/2} = \begin{pmatrix} a_+^{1/2} & 0 \\ \frac{1}{2\sqrt{a_+}} & a_+^{1/2} \end{pmatrix} \qquad \text{and} \qquad C^{-1/2} = \begin{pmatrix} a_+^{-1/2} & 0 \\ -\frac{1}{\left(2a_+^{3/2}\right)} & a_+^{-1/2} \end{pmatrix}.
$$

Then the four $2 \times 2$ blocks of the matrix $\exp(B_\lambda x)$ are

$$\cos(\lambda C^{-1/2}x) = \begin{pmatrix} \cos\frac{\lambda x}{\sqrt{a_+}} & 0 \\ \frac{\lambda x}{\left(2a_+^{3/2}\right)}\sin\frac{\lambda x}{\sqrt{a_+}} & \cos\frac{\lambda x}{\sqrt{a_+}} \end{pmatrix},$$

$\lambda^{-1}C^{1/2}\sin(\lambda C^{-1/2}x)$ equal to

$$\begin{pmatrix} \frac{a_+^{1/2}}{\lambda}\sin\frac{\lambda x}{\sqrt{a_+}} & 0 \\ \left[\frac{1}{2\lambda\sqrt{a_+}}\sin\frac{\lambda x}{\sqrt{a_+}} - \frac{x}{2a_+}\cos\frac{\lambda x}{\sqrt{a_+}}\right] & \frac{a_+^{1/2}}{\lambda}\sin\frac{\lambda x}{\sqrt{a_+}} \end{pmatrix}$$

and $-\lambda C^{-1/2}\sin(\lambda C^{-1/2}x)$ equal to

$$\begin{pmatrix} -\frac{\lambda}{a_+^{1/2}}\sin\frac{\lambda x}{\sqrt{a_+}} & 0 \\ \left[\frac{\lambda}{\left(2a_+^{3/2}\right)}\sin\frac{\lambda x}{\sqrt{a_+}} + \frac{\lambda^2 x}{2a_+^2}\cos\frac{\lambda x}{\sqrt{a_+}}\right] & -\frac{\lambda}{a_+^{1/2}}\sin\frac{\lambda x}{\sqrt{a_+}} \end{pmatrix}.$$

Let $\Phi(x)$ be a particular solution given as in (19), where the complex parameters $k_j$ are to be determined. Then

$$\Psi\Phi(0) = \begin{pmatrix} k_1v_1 + k_2v_2 \\ k_3v_3 + k_4v_4 \end{pmatrix}$$

and

$$\Psi\Phi(1) = \begin{pmatrix} \Psi\Phi(1)_1 \\ \Psi\Phi(1)_2 \end{pmatrix}$$

where

$$\Psi\Phi(1)_1 = k_1\left(v_1\cos\frac{\lambda}{\sqrt{a_+}} + \frac{v_2\lambda}{2a_+^{3/2}}\sin\frac{\lambda}{\sqrt{a_+}}\right) + k_2v_2\cos\frac{\lambda}{\sqrt{a_+}} +$$
$$+k_3\left(\frac{v_1\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}} + \frac{v_2}{2\lambda\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} - \frac{v_2}{2a_+}\cos\frac{\lambda}{\sqrt{a_+}}\right) + k_4\frac{v_2\sqrt{a_+}}{\lambda}\sin\frac{\lambda}{\sqrt{a_+}}$$

and

$$\Psi\Phi(1)_2 = k_1\left(-\frac{v_3\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} + \frac{v_4\lambda}{2a_+^{3/2}}\sin\frac{\lambda}{\sqrt{a_+}} + \frac{v_4\lambda^2}{2a_+}\cos\frac{\lambda}{\sqrt{a_+}}\right) +$$
$$-k_2\frac{v_4\lambda}{\sqrt{a_+}}\sin\frac{\lambda}{\sqrt{a_+}} + k_3\left(v_3\cos\frac{\lambda}{\sqrt{a_+}} + \frac{v_4\lambda}{2a_+^{3/2}}\sin\frac{\lambda}{\sqrt{a_+}}\right) + k_4v_4\cos\frac{\lambda}{\sqrt{a_+}}.$$

The solution $\Phi$ satisfies the boundary conditions (18), if and only if

$$\begin{cases} k_1v_1 + k_2v_2 = 0 \\ k_3v_3 + k_4v_4 = 0 \\ \Psi\Phi(1)_1 = 0 \\ \Psi\Phi(1)_2 = 0 \end{cases}$$

A rather long but straightforward computation shows that the determinant of this $4 \times 4$ system of linear equations in $k_j$ is $EV(\lambda)$. ∎

We show that $AD$ can have non-real eigenvalues even when the spectrum of $A$ is positive.

**Example 17.** Put

$$A := \begin{pmatrix} 2/5 + 3i/10 & 3/5 - 3i/10 \\ 3/20 + 3i/10 & 17/20 - 3i/10 \end{pmatrix}.$$

Then the eigenvalues of $A$ are $a_+ = 1$, $a_- = 1/4$, and the eigenvectors

$$v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_- = \begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

Thus

$$EV(x) = 4i(1 - \cos(x)\cos(2x)) = 4i(1 - 2\cos^3(x) + \cos(x))$$

so that $EV(\lambda) = 0$, if and only if

$$\cos(\lambda) = 1 \qquad \text{or} \qquad \cos(\lambda) = -1/2 \pm i/2.$$

Hence

$$\mathrm{Spec}(AD) = \{4k^2\pi^2, (\lambda_\pm + 2k\pi)^2\}_{k \in \mathbb{Z}}$$

where $\lambda_\pm = \arccos(-1/2 \pm i/2) \approx 2.02 \pm 0.53i$.

## 6. Real matrices

In this section we explore some connections between the entries of the matrix $A$ and the global behaviour of $\mathrm{Spec}(AD)$ when $A \in \mathbb{R}^{2\times 2}$. Alongside we discuss conditions to ensure similarity to a self-adjoint operator. For completeness of the picture, below and elsewhere we allow $\det(A) = 0$.

Our first task is to reduce to two parameters the four that are initially given as entries of $A$. This leads us to five different types of matrices to deal with. For $a, d \in \mathbb{R}$, let

$$A_0 := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad A_1 := \begin{pmatrix} a & 1 \\ 1 & d \end{pmatrix}, \quad A_2 := \begin{pmatrix} a & 0 \\ 1 & d \end{pmatrix},$$

$$A_3 := \begin{pmatrix} a & 1 \\ 0 & d \end{pmatrix} \qquad \text{and} \qquad A_4 := \begin{pmatrix} a & -1 \\ 1 & d \end{pmatrix}.$$

We show that the $A_j D$ generate any $AD$, $A \in \mathbb{R}^{2\times 2}$ via similarity transformations.

**Lemma 18.** *If $A \in \mathbb{R}^{2\times 2}$, then $AD$ is similar to $\alpha A_j D$ for some $\alpha, a, d \in \mathbb{R}$ and $j = 0, \dots, 4$.*

**Proof.** Let

$$A = \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix}.$$

If $bc = 0$, the proof is trivial. Let

$$A(r) := \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix} = \begin{pmatrix} \tilde{a} & r^{-1}b \\ rc & \tilde{d} \end{pmatrix}$$

Then, $AD$ is similar to $A(r)D$ for all $r \neq 0$. If $b/c > 0$,

$$A(\sqrt{b/c}) = \begin{pmatrix} \tilde{a} & \sqrt{bc} \\ \sqrt{bc} & \tilde{d} \end{pmatrix} = \alpha A_1$$

for $\alpha = \sqrt{bc}$, $a = \tilde{a}/\sqrt{bc}$ and $d = \tilde{d}/\sqrt{bc}$. If $b/c < 0$,

$$A(\sqrt{-b/c}) = \begin{pmatrix} \tilde{a} & \mp\sqrt{-bc} \\ \pm\sqrt{-bc} & \tilde{d} \end{pmatrix} = \pm\alpha A_4$$

for $\alpha = \sqrt{-bc}$, $a = \pm\tilde{a}/\sqrt{-bc}$ and $d = \pm\tilde{d}/\sqrt{-bc}$.            ∎

The case $j = 0$ was already described in corollary 3. Indeed if $ad \neq 0$ then $A_0 D$ is similar to a self-adjoint operator and

$$\operatorname{Spec}(A_0 D) = \{an^2\pi^2, dn^2\pi^2\}_{n=0}^{\infty} \subset \mathbb{R}.$$

## 6.1. Matrix $A_1$.

Since $a$ and $d$ are real, $A_1 = A_1^*$. Let $b_{\pm}$ be the eigenvalues of $A_1$. Then

$$b_{\pm} = \frac{a + d \pm \sqrt{(a-d)^2 + 4}}{2},$$

so that

   i) $b_+ \geq b_- > 0$, if and only if $ad > 1$ and $a, d > 0$,
   ii) $b_- \leq b_+ < 0$, if and only if $ad > 1$ and $a, d < 0$,
  iii) $b_+$ and $b_-$ have opposite signs, if and only if $ad < 1$.

**Theorem 19.** *The following statements are true.*

   a) *If $ad = 1$ then* $\operatorname{Spec}(A_1 D) = \mathbb{C}$.
   b) *If $ad > 1$ and $a, d > 0$ then $A_1 D$ is similar to a non-negative operator so that* $\operatorname{Spec}(A_1 D) \subset [0, \infty)$.
   c) *If $ad > 1$ and $a, d < 0$ then $-A_1 D$ is similar to a non-negative self-adjoint operator so that* $\operatorname{Spec}(A_1 D) \subset (-\infty, 0]$.
   d) *If $ad < 1$ then* $\operatorname{Spec}(A_1 D) \subset \mathbb{R}$.

**<u>Proof.</u>** If $ad = 1$, the matrix $A_1$ is singular so according to lemma 1, $A_1 D$ is not a closed operator. This shows a). Statement b) is consequence of i) and theorem 8, and statement c) is consequence of ii) and theorem 8.

Let us show d). For $\varepsilon \in \mathbb{R}$, let

$$B(\varepsilon) := A_1 + i\varepsilon.$$

Then

$$\operatorname{Num}(B(\varepsilon)) \subset \{\operatorname{Im}(z) > 0\} \qquad \varepsilon > 0$$

and

$$\operatorname{Num}(B(\varepsilon)) \subset \{\operatorname{Im}(z) < 0\} \qquad \varepsilon < 0.$$

According to theorem 10,

$$\mathrm{Spec}\,(B(\varepsilon)D) \subset \{\mathrm{Im}(z) \geq 0\} \qquad \varepsilon > 0$$

and

$$\mathrm{Spec}\,(B(\varepsilon)D) \subset \{\mathrm{Im}(z) \leq 0\} \qquad \varepsilon < 0.$$

Since $B(\varepsilon)D$ is a holomorphic family of type (A) in a neighbourhood of $\varepsilon = 0$ and $B(0) = A_1$, *a fortiori*

$$\mathrm{Spec}\,(A_1 D) \subset \mathbb{R}. \quad \blacksquare$$

Although $A_1 = A_1^*$, it is unclear to us whether $A_1 D$ is similar to self-adjoint in the latter case.

### 6.2. Matrices $A_2$ and $A_3$.

Since the results for the matrix $A_3$ are analogous and shown in a similar manner as for $A_2$, we will only consider the latter.

**Theorem 20.** *The following statements are true.*

a) *If $ad = 0$ then $\mathrm{Spec}\,(A_2 D) = \mathbb{C}$.*

b) *If $ad \neq 0$ then $\mathrm{Spec}\,(A_2 D) = \{a\pi^2 n^2, d\pi^2 n^2\}_{n=0}^{\infty}$.*

c) *If $ad > 0$, for all $\varepsilon > 0$ there exists $k_\varepsilon > 0$ independent of $z$, such that*

$$\|(A_2 D - z)^{-1}\| \leq \frac{k_\varepsilon}{|z|} \qquad z \notin \pm S(-\varepsilon, \varepsilon),$$

*where the symbol $\pm$ is chosen according to the symbol of $a$.*

d) *If $ad < 0$, then for all $\varepsilon > 0$ there exists $k_\varepsilon > 0$ independent of $z$, such that*

$$\|(A_2 D - z)^{-1}\| \leq \frac{k_\varepsilon}{|z|}$$

*for all $z \notin S(-\varepsilon, \varepsilon) \cup S(-\pi - \varepsilon, -\pi + \varepsilon)$.*

e) *If $a = d \neq 0$, let $\varepsilon > 0$ and $z_r = 4a\pi^2 r^2 \pm i\varepsilon$. Then there exists a constant $k_\varepsilon > 0$ independent of $r$, such that*

$$\|(A_2 D - z_r)^{-1}\| \geq k_\varepsilon |z_r|^{1/4}$$

*for all $r = 1, 2, \ldots$.*

**Proof.** If $ad = 0$, the matrix $A_2$ is singular so according to lemma 1, $A_2 D$ is not a closed operator. This shows a).

Let us show b). If $a \neq d$, the matrix $A_2$ is diagonalizable and

$$A_2 = \begin{pmatrix} a - d & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} (a-d)^{-1} & 0 \\ -(a-d)^{-1} & 1 \end{pmatrix}.$$

Then, according to theorem 15,

$$EV(x) = k_0 \sin\left(\frac{x}{\sqrt{a}}\right) \sin\left(\frac{x}{\sqrt{d}}\right)$$

where $k_0$ is constant in $x$. If $a = d$, $A_2$ is already in Jordan form and according to theorem 16,

$$EV(x) = - \left[ \sin \left( \frac{x}{\sqrt{a}} \right) \right]^2.$$

Hence in both cases

$$\mathrm{Spec}\,(A_2 D) = \{a\pi^2 n^2, d\pi^2 n^2\}_{n=0}^{\infty}.$$

Statements c) is consequence of corollary 11 and statement d) is consequence of corollary 14. For statement e) use theorem 12 and the fact that $|z_r|$ is of order $r^2$.     ∎

### 6.3. Matrix $A_4$.

Formally speaking, so far the spectrum of $A_j D$ for $j = 0, \ldots, 3$ reproduces the spectrum of $A_j$ in the following sense: if $A_j$ is non-degenerated and both eigenvalues of $A_j$ are positive (negative) then $\mathrm{Spec}\,(A_j D)$ is non-negative (non-positive), and if the eigenvalues are of opposite sign then $A_j D$ possess both positive and negative spectrum. There is no reason to expect the same for $j = 4$, in fact this case is less simple due to the way the entries of $A_4$ interact with the boundary conditions.

The eigenvalues of $A_4$ are given by

$$b_{\pm} := \frac{a + d \pm \sqrt{(a-d)^2 - 4}}{2}. \tag{20}$$

Then

i) $b_+ = b_-$, if and only if $|a - d| = 2$. In this case $A_4$ is not a diagonalizable matrix.

ii) $b_{\pm}$ are real and have opposite signs, if and only if $ad < -1$.

iii) $b_+ > b_- > 0$, if and only if $ad > -1$, $|a - d| > 2$ and $a + d > 0$.

iv) $b_- < b_+ < 0$, if and only if $ad > -1$, $|a - d| > 2$ and $a + d < 0$.

v) $b_{\pm}$ are non-real with $b_+ = \overline{b_-}$, if and only if $|a - d| < 2$.

vi) $A_4$ is singular, if and only if $ad = -1$.

Motivated by this and for simplicity, we can divide the plane into 6 disjoint regions $R_k$,

$$
\begin{aligned}
R_1 &:= \{(a, d) \in \mathbb{R}^2 : |a - d| = 2, a \neq \pm 1\}, \\
R_2 &:= \{(a, d) \in \mathbb{R}^2 : ad < -1\}, \\
R_3 &:= \{(a, d) \in \mathbb{R}^2 : ad > -1, |a - d| > 2, a + d > 0\}, \\
R_4 &:= \{(a, d) \in \mathbb{R}^2 : ad > -1, |a - d| > 2, a + d < 0\}, \\
R_5 &:= \{(a, d) \in \mathbb{R}^2 : |a - d| < 2\}, \\
R_6 &:= \{(a, d) \in \mathbb{R}^2 : ad = -1\}.
\end{aligned}
$$

Clearly $\mathbb{R}^2 = \bigcup R_k$. Below we establish the spectral results for $A_4 D$ separately in each region $R_k$.
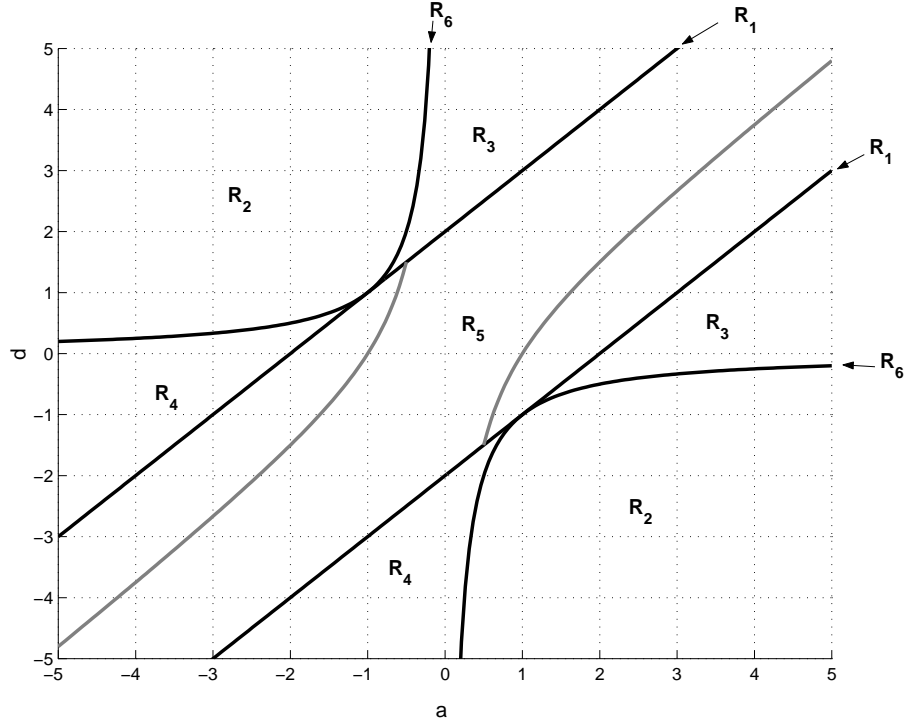
FIGURE 1. Different regions of the plane in which $\mathrm{Spec}(A_4 D)$ exhibits a similar behaviour. The grey line is $\{a^2 - ad - 1 = 0\} \cap R_5$. See theorems 21-27.

Two cases are similar to what we have found so far.

**Theorem 21.** *The following statements are true.*

a) *If $(a, d) \in R_6$, then $\mathrm{Spec}(A_4 D) = \mathbb{C}$.*

b) *If $(a, d) \in R_2$, then $\mathrm{Spec}(A_4 D) \subset \mathbb{R}$ and $A_4 D$ is similar to a self-adjoint operator whose numerical range is the whole real line.*

**<u>Proof.</u>** If $ad = -1$, the matrix $A_4$ is singular so according to lemma 1, $A_4 D$ is not a closed operator. This shows a).

Let us show b). Let $J$ be as in section 4. Then

$$A_4 D = (A_4 J)(JD) = \begin{pmatrix} a & 1 \\ 1 & -d \end{pmatrix} \tilde{D} = \tilde{A}\tilde{D}.$$

Here $\tilde{A} = \tilde{A}^*$ and the eigenvalues of $\tilde{A}$ are

$$\tilde{b}_\pm = \frac{a - d \pm \sqrt{(a+d)^2 + 4}}{2}.$$

Since $ad < -1$, $\tilde{b}_\pm$ are either both positive or both negative. If they are both positive, $\tilde{A} > 0$ so that theorem 8-b) provides the desired

conclusion. If they are both negative apply the above argument to $-A_4D$. ∎

In order to find $\mathrm{Spec}(A_4D)$ in $R_k$ for $k = 1, 3, 4, 5$, we ought to rely on properties of the transcendental function $EV(x)$. Nonetheless, theorem 22 provides some indication of what we should expect, it bases on the observation that if both $a$ and $d$ are positive,

$$A_4 + A_4^* = \begin{pmatrix} 2a & 0 \\ 0 & 2d \end{pmatrix} > 0,$$

so by virtue of lemma 9, $\mathrm{Spec}(A_4D) \subset \{\mathrm{Re}(z) \geq 0\}$.

**Theorem 22.** *If both $a$ and $d$ are positive, then*

$$\mathrm{Spec}(A_4D) \subset S(-\omega, \omega)$$

*where $\sin \omega = 1/\sqrt{ad+1}$ for $0 < \omega < \pi/2$.*

**Proof.** The numerical range of $A_4$ is an ellipse whose foci are $b_\pm$ and largest diameter is of length $|a - d|$. It is easy to see that $S(-\omega, \omega)$ is the minimal sector that contains such an ellipse. Use theorem 10 to complete the proof. ∎

Since

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a & -1 \\ 1 & -d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = - \begin{pmatrix} a & -1 \\ 1 & d \end{pmatrix}$$

and because of diagonal matrices commute with the boundary conditions, $\mathrm{Spec}(A_4D) \subset -S(-\omega, \omega)$ where both $a$ and $d$ are negative. This also shows that the spectral results for $A_4D$ are symmetric with respect to the transformation $(a, d) \mapsto (-a, -d)$. Below we will employ this symmetry often without mention.

In order to describe $\mathrm{Spec}(A_4D)$ in $R_5$, we will make use of the following technical result.

**Lemma 23.** *Let $\alpha \in \mathbb{C}$ be such that $\mathrm{Re}(\alpha^2) \geq 0$, let $-1 \leq c \leq 1$ and let*

$$F(x) := 1 - \cos(\alpha x) \cos(\overline{\alpha}x) - c \sin(\alpha x) \sin(\overline{\alpha}x) \qquad x \in \mathbb{C}.$$

*Then $F(x)$ has an infinite number of zeros in the complex plane and*

   a) *if $c = -1$, then $F(x) = 0$, if and only if $\sin(\mathrm{Re}(\alpha)\,x) = 0$,*
   b) *if $c = 1$, then $F(x) = 0$, if and only if $\sinh(\mathrm{Im}(\alpha)x) = 0$,*
   c) *if $-1 < c < 1$, then $F(x)$ only has a finite number of zeros lying on the real and imaginary axis.*

**Proof.** Let $\alpha =: \rho + i\mu$ so that $\rho \geq \mu > 0$ and let $x =: x_1 + ix_2$ for $x_1, x_2 \in \mathbb{R}$.

In order to show a), assume $c = -1$. Then

$$
\begin{aligned}
|F(x)|^2 &= |1 - \cos[\alpha(x_1 + ix_2)]\cos[\overline{\alpha}(x_1 + ix_2)]+ \\
&\qquad + \sin[\alpha(x_1 + ix_2)]\sin[\overline{\alpha}(x_1 + ix_2)]|^2 \\
&= 4[\cos^2(\rho x_1) - \cosh^2(\rho x_2)]^2.
\end{aligned}
$$

Hence

$$
F(x) = 0,
$$

if and only if $\cosh(\rho x_2) = 1$ and $\cos(\rho x_1) = \pm 1$. This gives a).

Similarly for b), assume $c = 1$. Then

$$
\begin{aligned}
|F(x)|^2 &= |1 - \cos[\alpha(x_1 + ix_2)]\cos[\overline{\alpha}(x_1 + ix_2)]+ \\
&\qquad - \sin[\alpha(x_1 + ix_2)]\sin[\overline{\alpha}(x_1 + ix_2)]|^2 \\
&= 4[\cosh^2(\mu x_1) - \cos^2(\mu x_2)]^2.
\end{aligned}
$$

Hence

$$
F(x) = 0,
$$

if and only if $\cosh(\mu x_1) = 1$ and $\cos(\mu x_2) = \pm 1$.

Let us show assertion c). If $x \in \mathbb{R}$, then

$$
\begin{aligned}
F(x) &= 1 - \cos(\alpha x)\overline{\cos(\alpha x)} - c\sin(\alpha x)\overline{\sin(\alpha x)} \\
&= 1 - |\cos(\alpha x)|^2 - c|\sin(\alpha x)|^2 \\
&= 1 - \cos^2(\rho x) - c\sin^2(\rho x) - (1 + c)\sinh^2(\mu x)
\end{aligned}
$$

and

$$
\begin{aligned}
F(ix) &= 1 - \cos(-i\overline{\overline{\alpha}}x)\cos(i\overline{\alpha}x) - c\sin(-i\overline{\overline{\alpha}}x)\sin(i\overline{\alpha}x) \\
&= 1 - \overline{\cos(i\overline{\alpha}x)}\cos(i\overline{\alpha}x) + c\overline{\sin(i\overline{\alpha}x)}\sin(i\overline{\alpha}x) \\
&= 1 - |\cos(i\overline{\alpha}x)|^2 + c|\sin(i\overline{\alpha}x)|^2 \\
&= 1 - \cos^2(\mu x) + c\sin^2(\mu x) - (1 - c)\sinh^2(\rho x).
\end{aligned}
$$

Hence, if $-1 < c < 1$,

$$
\lim_{x \to \pm\infty} F(x) = -\infty \qquad \text{and} \qquad \lim_{x \to \pm\infty} F(ix) = -\infty.
$$

Since $F(x)$ is a smooth function, c) follows.

Finally let us show that $F(x)$ has a infinite number of zeros. Suppose that $F$ only has a finite number of zeros $0, z_1, \ldots, z_n$ where the $z_j$ repeats as many times as its order. Then

$$
G(x) = \frac{F(x)}{x^2 \prod_{j=1}^{n}(x - z_j)}
$$

is an entire function with no zeros. By virtue of the Weierstrass factorization theorem, there is an entire function $g(x)$ such that $G(x) = e^{g(x)}$. Then

$$
F(x) = \left[ x^2 \prod_{j=1}^{n}(x - z_j) \right] e^{g(x)} =: p(x)e^{g(x)}.
$$

Since it is a combination of sines and cosines, the order (cf. [3, p.285]) in the sense of entire functions of $F(x)$ is $\lambda = 1$. Thus by virtue of Hadamard's factorization theorem, $g(x)$ is a polynomial of degree 1 in $x$ and so

$$F(x) = p(x)e^{kx+l}$$

for suitable $k, l \in \mathbb{C}$. Since $p(x)$ is a polynomial, this is clearly a contradiction, so $F(x)$ must have an infinite number of zeros. ■

**Theorem 24.** *Let $(a, d) \in R_5$.*

a) *If $(a, d) \in \{a^2 - ad - 1 = 0\} \cap \{-2 < a - d < 0\}$, then*

$$\mathrm{Spec}\,(A_4 D) = \left\{-k^2\pi^2/[\mathrm{Im}(b_+^{-1/2})]^2\right\}_{k\in\mathbb{Z}} \subset (-\infty, 0].$$

b) *If $(a, d) \in \{a^2 - ad - 1 = 0\} \cap \{0 < a - d < 2\}$, then*

$$\mathrm{Spec}\,(A_4 D) = \left\{k^2\pi^2/[\mathrm{Re}(b_+^{-1/2})]^2\right\}_{k\in\mathbb{Z}} \subset [0, \infty).$$

c) *If $(a, d) \notin \{a^2 - ad - 1 = 0\}$, then $\mathrm{Spec}\,(A_4 D)$ is infinite but it only intersects the real line in a finite number of points.*

**Proof.** By virtue of v), $A_4$ is diagonalizable. We assume $a + d \geq 0$, so that

$$\{b_\pm\} \subset \{\mathrm{Re}(z) \geq 0\}.$$

Let

$$y := \sqrt{4 - (a - d)^2} \qquad \text{and} \qquad \gamma_\pm = a - d \pm iy.$$

Then

$$A_4 = \begin{pmatrix} \gamma_+ & \gamma_- \\ 2 & 2 \end{pmatrix} \begin{pmatrix} b_+ & 0 \\ 0 & b_- \end{pmatrix} \begin{pmatrix} \frac{1}{2iy} & -\frac{\gamma_-}{4iy} \\ -\frac{1}{2iy} & \frac{\gamma_+}{4iy} \end{pmatrix}.$$

Let $\vartheta := \arg \gamma_+$ and $\alpha := 1/\sqrt{b_+}$ so that $\overline{\alpha} = 1/\sqrt{b_-}$. Then

$$\frac{EV(x)}{4\gamma_-^2} = \frac{2\gamma_+}{\gamma_-}[1 - \cos(\alpha x)\cos(\overline{\alpha}x)] - \left(\frac{\gamma_+^2}{\gamma_-^2}\sqrt{\frac{b_+}{b_-}} + \sqrt{\frac{b_-}{b_+}}\right)\sin(\alpha x)\sin(\overline{\alpha}x)$$

$$= \frac{2\gamma_+}{\gamma_+}[1 - \cos(\alpha x)\cos(\overline{\alpha}x)] - \left(\frac{\gamma_+^2}{\gamma_+^2}\frac{\overline{\alpha}}{\alpha} + \frac{\alpha}{\overline{\alpha}}\right)\sin(\alpha x)\sin(\overline{\alpha}x)$$

$$= 2e^{i2\vartheta}[1 - \cos(\alpha x)\cos(\overline{\alpha}x)] - \left(e^{i4\vartheta}\frac{\overline{\alpha}}{\alpha} + \frac{\alpha}{\overline{\alpha}}\right)\sin(\alpha x)\sin(\overline{\alpha}x)$$

$$= 2e^{i2\vartheta}[1 - \cos(\alpha x)\cos(\overline{\alpha}x) - c\sin(\alpha x)\sin(\overline{\alpha}x)]$$

$$= 2e^{i2\vartheta}F(x),$$

$$(21)$$

where $F(x)$ and

$$c := \frac{(\overline{\alpha}e^{i2\vartheta}/\alpha) + (\alpha e^{-i2\vartheta}/\overline{\alpha})}{2}$$

$$= \frac{e^{i(2\vartheta - 2\arg(\alpha))} + e^{-i(2\vartheta - 2\arg(\alpha))}}{2}$$

$$= \cos(2\vartheta - 2\arg(\alpha)) = \cos(2\vartheta + \arg b_+)$$

are as in lemma 23.

Let us show a). The hypothesis $a - d < 0$ ensures $-1 < c \leq 1$. Furthermore $c = 1$, if and only if

$$\frac{\mathrm{Im}(\gamma_+^2)}{\mathrm{Re}(\gamma_+^2)} = -\frac{\mathrm{Im}(b_+)}{\mathrm{Re}(b_+)}.$$

The latter occurs, if and only if

$$\frac{y(a - d)}{(a - d)^2 - 2} = -\frac{y}{a + d}.$$

By simplifying this identity, we gather that $c = 1$ for $a^2 - ad - 1 = 0$ which is precisely our assumption. Then, lemma 23-b) and (21) complete the proof of a).

For b), notice that since $a - d > 0$, the constant $c$ is now such that $-1 \leq c < 1$ and $c = -1$, if and only if

$$\frac{\mathrm{Im}(\gamma_+^2)}{\mathrm{Re}(\gamma_+^2)} = -\frac{\mathrm{Im}(b_+)}{\mathrm{Re}(b_+)}.$$

Therefore a similar argument as for a) and lemma 23-a) show this case. In order to prove c) use the fact that $-1 < c < 1$ in

$$R_5 \setminus \{a^2 - ad - 1 = 0\},$$

lemma 23-c) and (21). ∎

**Theorem 25.** *In the regions $R_3$ and $R_4$, $\mathrm{Spec}(A_4 D)$ is infinite, and*

$$\mathrm{Spec}(A_4 D) \subset \{(r + iy_0)^2 : r \in \mathbb{R}\} + [0, \infty) \qquad (a, d) \in R_3$$

$$\mathrm{Spec}(A_4 D) \subset \{-(r + iy_0)^2 : r \in \mathbb{R}\} + (-\infty, 0] \qquad (a, d) \in R_4,$$

*where in both cases the constant $y_0 > 0$ only depends upon $(a, d)$.*

**Proof.** We show the result only for $R_3$. According to iii), in this case $0 < b_- < b_+$ and $A_4$ is diagonalizable. By expressing the trigonometric functions in exponential form,

$$EV(x) = k_1 - k_1 \cos(\alpha x)\cos(\beta x) - k_2 \sin(\alpha x)\sin(\beta x)$$

$$= k_1 + \frac{k_2 - k_1}{4}\left[e^{i(\alpha + \beta)x} + e^{-i(\alpha + \beta)x}\right] +$$

$$- \frac{k_2 + k_1}{4}\left[e^{i(\alpha - \beta)x} + e^{-i(\alpha - \beta)x}\right]$$

where $k_1, k_2 \in \mathbb{R}$ and $0 < \beta < \alpha$ are constants we do not need to specify here. A similar argument involving Hadamard's theorem as in the proof of lemma 23 shows that $\mathrm{Spec}(A_4 D)$ is infinite.

By putting $x = r + iy$ where $r, y \in \mathbb{R}$, $\gamma := \alpha + \beta > 0$ and $\delta := \alpha - \beta > 0$,

$$EV(r + iy) = k_1 + \frac{k_2 - k_1}{4} \left[ \mathrm{e}^{-\gamma y} \mathrm{e}^{i\gamma r} + \mathrm{e}^{\gamma y} \mathrm{e}^{-i\gamma r} \right] +$$
$$- \frac{k_2 + k_1}{4} \left[ \mathrm{e}^{-\delta y} \mathrm{e}^{i\delta r} + \mathrm{e}^{\delta y} \mathrm{e}^{-i\delta r} \right].$$

Since $\gamma > \delta > 0$, if we chose $y >> 0$, the term $\mathrm{e}^{\gamma y}$ dominates the expression and so $|EV(r + iy)| \geq c > 0$ for a suitable $c$ independent of $r$. If we chose $y << 0$, the term $\mathrm{e}^{-\gamma y}$ is the one that dominates and again $|EV(r + iy)|$ is large. This shows that all the zeros of $EV(x)$ must be contained in a band $\{-y_0 \leq \mathrm{Im}(x) \leq y_0\}$. ∎

The above theorem does not rule out the possibility of negative eigenvalues when $ad < 0$. We will see in the numerical examples, evidence of points in this region such that $A_4 D$ has indeed negative spectrum.

With regard to finding the minimal $y_0$. We will see in section 7 an argument involving Chebyshev polynomial that allows us to compute in closed form $\mathrm{Spec}(A_4 D)$ for a certain dense subset of $R_3$. We will also illustrate this technique in various examples where the parabolic region is found explicitly.

If $(a, d) \in R_1$, the matrix $A_4$ is not diagonalizable and so $EV(x)$ is given by theorem 16 instead of theorem 15. Nevertheless, similar techniques to the ones we have seen so far apply to this case.

**Lemma 26.** *Let $0 \neq c \in \mathbb{R}$ and let*

$$F(x) = x^2 + c[\sin(x)]^2 \qquad x \in \mathbb{C}.$$

*Then $F(x)$ has an infinite number of zeros in the complex plane but only a finite number of them lie on $\mathbb{R}$ and on $i\mathbb{R}$.*

**Proof.** See the proofs of lemma 23 and theorem 25. ∎

**Theorem 27.** *Let $(a, d) \in R_1$. If $(a, d) = (\pm 1/2, \mp 3/2)$, then $\mathrm{Spec}(A_4 D) = \{0\}$. Otherwise $\mathrm{Spec}(A_4 D)$ is infinite but it only intersects the real line in a finite number of points.*

**Proof.** If $a - d = 2$,

$$A_4 = \begin{pmatrix} b_+ + 1 & -1 \\ 1 & b_+ - 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_+ & 0 \\ 1 & b_+ \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and if $a - d = -2$,

$$A_4 = \begin{pmatrix} b_+ - 1 & -1 \\ 1 & b_+ + 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_+ & 0 \\ 1 & b_+ \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$EV(x) = \frac{x^2}{4b_+^3} - \left(1 \pm \frac{1}{2b_+}\right)^2 \left[\sin\left(\frac{x}{\sqrt{b_+}}\right)\right]^2, \qquad a - d = \pm 2.$$

The first statement follows from the fact that if $(a, d) = (\pm 1/2, \mp 3/2)$, then $b_+ = \mp 1/2$ and so the trigonometric term disappear. The second follows from lemma 26. $\blacksquare$

Notice that the curve $a^2 - ad - 1 = 0$ meets the region $R_1$ at $(\pm 1/2, \mp 3/2)$. These are the only points where $\mathrm{Spec}(A_4 D)$ is finite. Since all self-adjoint operators with compact resolvent must have an infinite number of eigenvalues, $A_4 D$ is not similar to self-adjoint. All this suggests that for $(a, d)$ in a small neighbourhood of these points, $\mathrm{Spec}(A_4 D)$ must be highly unstable. In the next section we explore closely this idea.

## 7. Some numerical results

As mentioned previously, this section is devoted to investigating some aspects of the global spectral evolution of $AD$ when we move the entries of the matrix $A$. To be more precise, we consider $A = A_4$ (see section 6) and compute $\mathrm{Spec}(A_4 D)$ as $(a, d)$ moves along various lines inside $R_1 \cup R_3 \cup R_5 \subset \mathbb{R}^2$. We also introduce a technique that allows us to find explicitly $\mathrm{Spec}(A_4 D)$ when $(a, d)$ are in a certain dense subset of $R_3$ by computing the roots of certain polynomial $G(w)$.

Our first task is to decompose $R_3$ into a disjoint union of curves in order to find the dense subset. For $\alpha > 1$, let

$$d_\pm(a) := \frac{a(\alpha^4 + 1) \pm \sqrt{(\alpha^4 - 1)^2 a^2 + 4\alpha^2(\alpha^2 + 1)^2}}{2\alpha^2}$$

and let

$$\Lambda_\pm(\alpha) := \{(a, d_\pm(a)) : a > \mp 1\}.$$

Then

$$R_3 \cap \{a - d < 0\} = \bigcup_{\alpha > 1} \Lambda_+(\alpha) \qquad \text{and} \qquad R_3 \cap \{a - d > 0\} = \bigcup_{\alpha > 1} \Lambda_-(\alpha).$$

The motivation for this decomposition is found by observing that for

$$A_4 = \begin{pmatrix} a & -1 \\ 1 & d_\pm(a) \end{pmatrix},$$

$\sqrt{b_+/b_-} = \alpha$, where $0 < b_- < b_+$ are the eigenvalues of $A_4$. That is, $\Lambda_\pm$ are level curves of $\sqrt{b_+/b_-}$ in the $(a, d)$-plane. Notice that

$$R_3 = \overline{\bigcup_{1 < \alpha \in \mathbb{Q}} \Lambda_+(\alpha) \cup \Lambda_-(\alpha)}.$$

The key idea behind finding $G(w)$ is that for $(a, d) \in \Lambda_\pm(\alpha)$ where $1 < \alpha \in \mathbb{Q}$, the zeros of the transcendental function are periodic in the

horizontal direction. We show how to construct this polynomial. The transcendental function for $A_4 D$ is

$$EV(x) = k_1 \left[ 1 - \cos\left(\frac{x}{\sqrt{b_+}}\right) \cos\left(\frac{x}{\sqrt{b_-}}\right) \right] +$$

$$- k_2 \sin\left(\frac{x}{\sqrt{b_+}}\right) \sin\left(\frac{x}{\sqrt{b_-}}\right)$$

where $k_1$ and $k_2$ are two real constants depending upon $a$ and $d$ which we do not need to specify here. Since

$$\sqrt{b_+/b_-} = \alpha = p/q, \qquad p, q \in \mathbb{Z}^+,$$

$\sqrt{b_\pm}$ are rationally related and so the zeros of $EV(x)$ appear periodically in lines parallel to the real axis. By putting $z = x/(q\sqrt{b_+})$,

$$EV(x) = k_1[1 - \cos(pz)\cos(qz)] - k_2 \sin(pz)\sin(qz)$$

$$= k_1 + \frac{k_2 - k_1}{2}\cos[(p+q)z] - \frac{k_2 + k_1}{2}\cos[(p-q)z],$$

where $p - q < p + q \in \mathbb{Z}^+$. Standard computations show that,

$$\cos(mz) = T_m(\cos(z)) \qquad\qquad m = 1, 2, \ldots$$

where $T_m$ a polynomial of degree $m$ (the $m^{th}$ Chebyshev polynomial of first order). Then by letting

$$G(w) := k_1 + \frac{k_2 - k_1}{2}T_{(p+q)}(w) - \frac{k_2 + k_1}{2}T_{(p-q)}(w),$$

$EV(x) = 0$, if and only if $G(\cos(z)) = 0$. Hence all the zeros of $EV(x)$ are of the form

$$(\pm \arccos(w_0) + 2n\pi)q\sqrt{b_+} \in \mathbb{C}, \qquad\qquad n \in \mathbb{Z}$$

where $w_0$ is a root of $G(w)$. In this manner, $\mathrm{Spec}(A_4 D)$ is generated by translations of the roots of $G(w)$.

Although the above method computes $\mathrm{Spec}(A_4 D)$ explicitly for $(a, d) \in \Lambda_\pm(\alpha), 1 < \alpha \in \mathbb{Q}$, its numerical implementation for large $p + q$ ($> 20$ in a PC) is highly unstable due to the well known instability of the roots of polynomials of high degree. Nevertheless, no other procedure tried so far, has proven to be more efficient for estimating large eigenvalues in $R_3$. Figures 2, 5 and 6 below were produced via this approach.

## 7.1. Spectral behaviour of $A_4$ for $(a, d)$ close to $(-1/2, 3/2)$.
By virtue of theorem 27, $\mathrm{Spec}(A_4 D) = \{0\}$ for $(a, d) = (-1/2, 3/2)$. In any small neighbourhood of this point, the spectrum of $A_4 D$ is infinite so high instability is to be expected. Since $A_4 D$ is holomorphic in $a$ and $d$, every non-zero eigenvalue of $A_4 D$ either concentrates at zero or diverges to $\infty$ for $(a, d) \to (-1/2, 3/2)$. We explore this phenomenon in some detail.
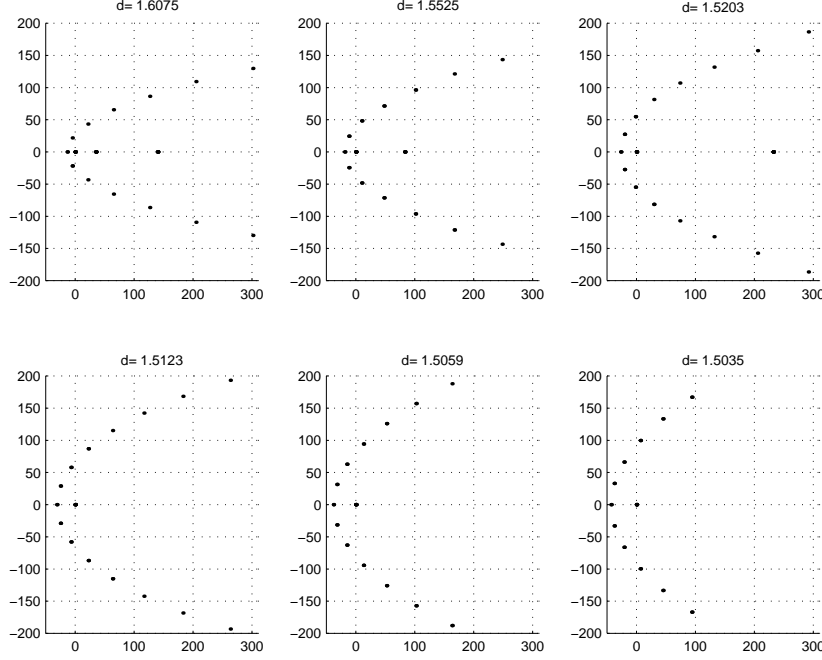
FIGURE 2. Evolution of the first 16 eigenvalues of $A_4D$ for $a = -1/2$ fixed and different values of $d$ close to $3/2$.

According to theorem 24-a), if $(a, d) \in R_5$ satisfy $a^2 - ad - 1 = 0$ and $-2 < a - d < 0$,

$$\text{Spec}(A_4D) = \left\{-k^2\pi^2/[\text{Im}(b_+^{-1/2})]^2\right\}_{k\in\mathbb{Z}},$$

where $b_+$ as in section 6.3. By taking $a \to -1/2$ and $d \to 3/2$,

$$b_+ = \frac{a + d + \sqrt{(a-d)^2 - 4}}{2} \to 1/2 \in \mathbb{R},$$

so that $\text{Im}(b_+^{-1/2}) \to 0$. Hence, all non-zero eigenvalues of $A_4D$ remain negative and escape to $-\infty$ as $(a, d) \in R_5$ approach the critical point on the curve $a^2 - ad - 1 = 0$.

In general, not every eigenvalue of $A_4D$ need to be in the left hand plane when $(a, d)$ is close to $(-1/2, 3/2)$. In figure 2 we consider the evolution of the first 16 eigenvalues of $A_4D$ for $a = -1/2$ fixed and 6 different values of $d$ from $d = 1.6075$ to $1.5035$. The awkward choice of $d$ correspond to the sensible values of $\alpha \in \mathbb{Q}$; each pair $(-1/2, d) \in \Lambda_+(\alpha)$ for $\alpha = 2, 8/5, 4/3, 5/4, 7/6, 9/8$. Notice that for large $p, q$ the polynomial $G(w)$ has $p + q$ roots and nonetheless all these roots but 0 lie on the same curve. This curve moves away from the origin and there
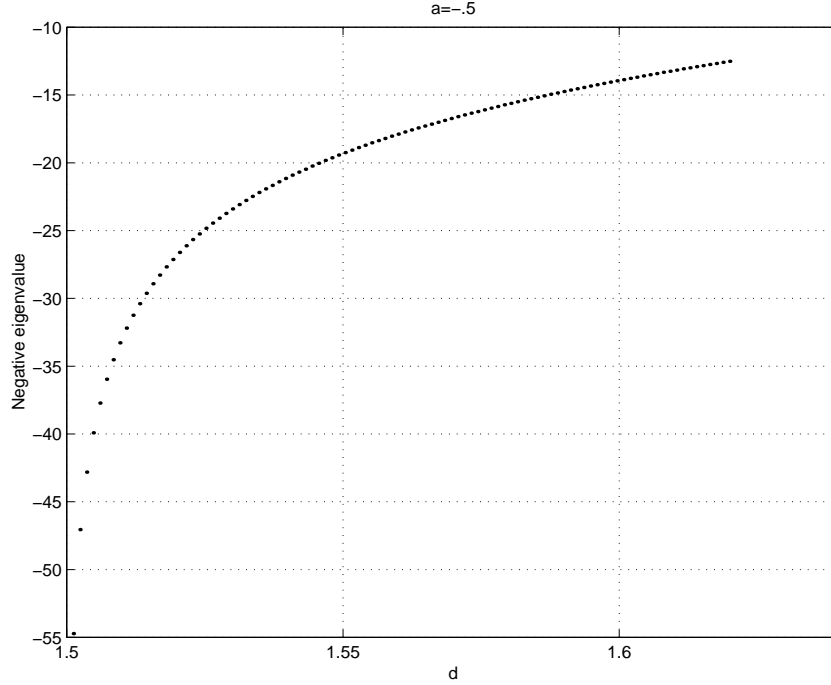
FIGURE 3. Evolution of the negative eigenvalue for $a = -1/2$ and 100 different values of $d$ linearly distributed on the segment $[1.5012, 1.6200]$.

is always a negative eigenvalue. The positive eigenvalues also escape rapidly to $+\infty$ and there are infinitely many of them.

In figure 3 we isolate the negative eigenvalue for $a = -1/2$ against 100 different values of $d$ close to $d = 3/2$. This provides indication of how rapidly it escapes to $-\infty$. In order to produce this picture, we made use of the algorithm that Matlab provides to find the zero of $EV(x)$ for $x$ on the imaginary axis. Comparing with the comment we made earlier in section 6.3, this provides points in $R_3$ such that $A_4 D$ has a negative eigenvalue of arbitrarily large modulus.

### 7.2. Non-real eigenvalues in $R_1$.

We now explore the transition from real to non-real spectrum by considering the spectral evolution of $A_4$ on the line

$$\{(0, d) \in R_3 \,:\, d > 2\}$$

close to $(0, 2) \in R_1$. In figure 4 we show the first 23 eigenvalues of $A_4 D$ for $a = 0$ and $d = 2$. We produced this graphic by reducing the equation $EV(x) = 0$ to a single real variable and then making use of the algorithm that Maple provides to find zeros of real functions. According to theorem 27, we know that $\operatorname{Spec}(A_4 D)$ is infinite but there is only
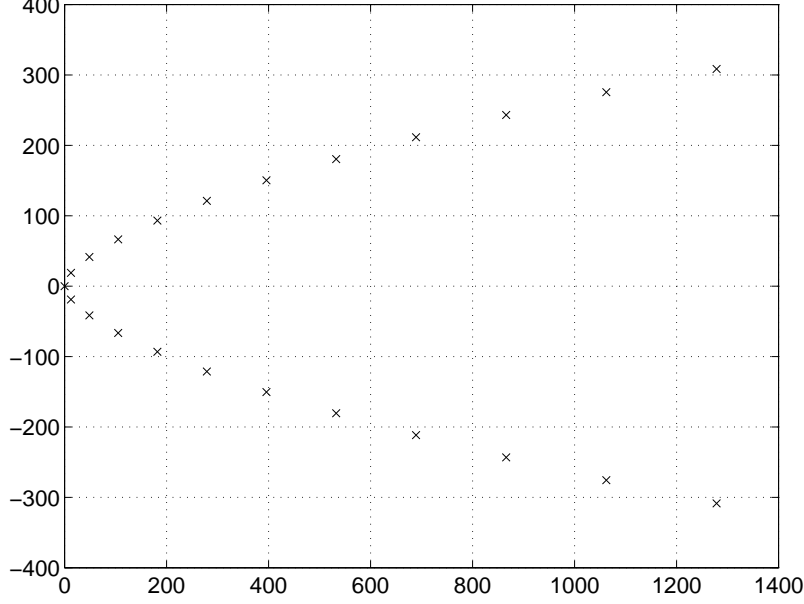
FIGURE 4. First 23 eigenvalues of $A_4 D$ for $a = 0$ and $d = 2$.

finite intersection with the real line. As the picture suggests, in this case the origin seems to be the only real eigenvalue.

Figure 5 shows the evolution of the first 17 eigenvalues (counting multiplicity) of $A_4$ when $a = 0$ for various different values of $d$ from $d = 3.3333$ to $2.0139$. Each pair $(0, d) \in \Lambda_+(\alpha)$ respectively for $\alpha = 3$, $5/2$, $9/4$, $2$, $9/5$, $3/2$, $5/4$, $9/8$. The numerical evidence suggests that for $d = 3.3333$ the spectrum is close to the real line and each eigenvalue is of multiplicity 2. Each of these operators has infinitely many real eigenvalues. Unfortunately the method we employed to find the roots of $G(w)$, is unable to deal with a finer partition of the $d$-interval. Nonetheless, the global behaviour of the spectrum can be appreciated, as $d$ approaches to 2, each real eigenvalue eventually splits into two conjugate non-real single eigenvalues stabilizing close to the region in figure 4 (see the step $d = 2.0139$). Notice that there is no spectrum in the left hand plane and compare with theorem 22.

### 7.3. Spectral evolution close to $R_6$.

Another type of peculiar behaviour can be observed as $(a, d) \in R_3$ approach the region $R_6$, where the matrix $A_4$ is singular and $\mathrm{Spec}(A_4 D) = \mathbb{C}$. Here we concentrate on the point $(-1, 1) \in R_6$.

Figure 6 shows the evolution of the first 100 eigenvalues of $A_4 D$ (represented by dots) as $(a, d) \in \Lambda_+(2)$ approaches to $(-1, 1) \in R_6$. Alongside we also picture the remaining eigenvalues (represented by crosses) that lie on the box $[0, 2000] \times [-300, 300]$. A very similar behaviour occurs for $(a, d) \in \Lambda_\pm(\alpha)$ as $(a, d) \to (\mp 1, \pm 1) \in R_6$ for

other values of $\alpha \in \mathbb{Q}$. It can not be appreciated in the graph provided but there are two conjugate eigenvalues whose real part is negative. These eigenvalues approach to the origin as $(a,d) \to (-1,1)$. All the remaining spectrum concentrates on the real line suggesting that $\mathrm{Spec}(A_4 D) \to [0,\infty)$ as $(a,d) \to (-1,1)$ this is in contrast with the fact that $\mathrm{Spec}(AD) = \mathbb{C}$ at $(-1,1)$.

Here we have chosen $p = 2$ and $q = 1$. This means that $G(w)$ is only of order 3 and so the spectrum is always generated by 3 points. It is not difficult to show analytically that all three roots converge to 0 and then rigorously prove that $\mathrm{Spec}(A_4 D) \to [0,\infty)$.

## References

[1] A. ASLANYAN, E. B. DAVIES, Spectral instability for some Schrödinger operators, *Numer. Math.* **85**(2000), 525-552.

[2] L. BOULTON, Non-self-adjoint harmonic oscillator, compact semigroups and pseudospectra, preprint, King's College London, 1999, $maths.SP/9909179$.

[3] J. B. CONWAY, *Functions of one complex variable I*, 2nd edition, Springer-Verlag, New York, 1978.

[4] E. B. DAVIES, Pseudospectra, the harmonic oscillator and complex resonances, *Proc. Roy. Soc. London A* **455**(1999), 585-599.

[5] E. B. DAVIES, Semi-classical states for non-self-adjoint Schrödinger operators, *Commun. Math. Phys.* **200**(1999), 35-41.

[6] T. KATO, *Perturbation theory for linear operators*, 2nd edition, Springer-Verlag, Berlin, 1980.

[7] M. REED, B. SIMON, *Methods of modern mathematical physics, volume 4: analysis of operators*, Academic press, New York, 1978.

[8] R. F. STREATER, Stability of a hot Smolochowski fluid, *Open Systems and Information Dynamics* **7**(2000), 1-9.

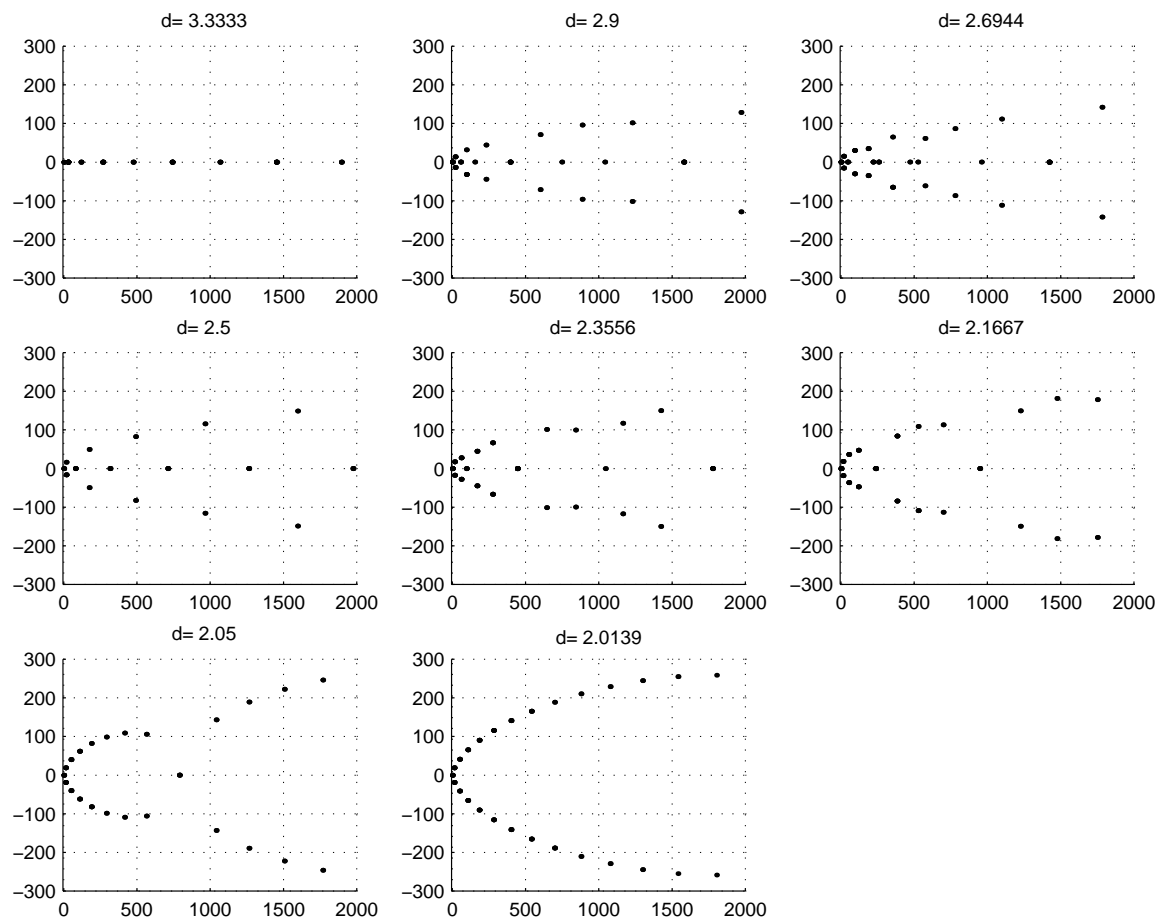[9] L. N. TREFETHEN, Pseudospectra of linear operators, *SIAM Review* **39**(1997), 383-406.

FIGURE 5. Evolution of the first 17 eigenvalues (counting multiplicity) of $A_4D$ for $a = 0$ and $d > 2$ close to $d = 2$.
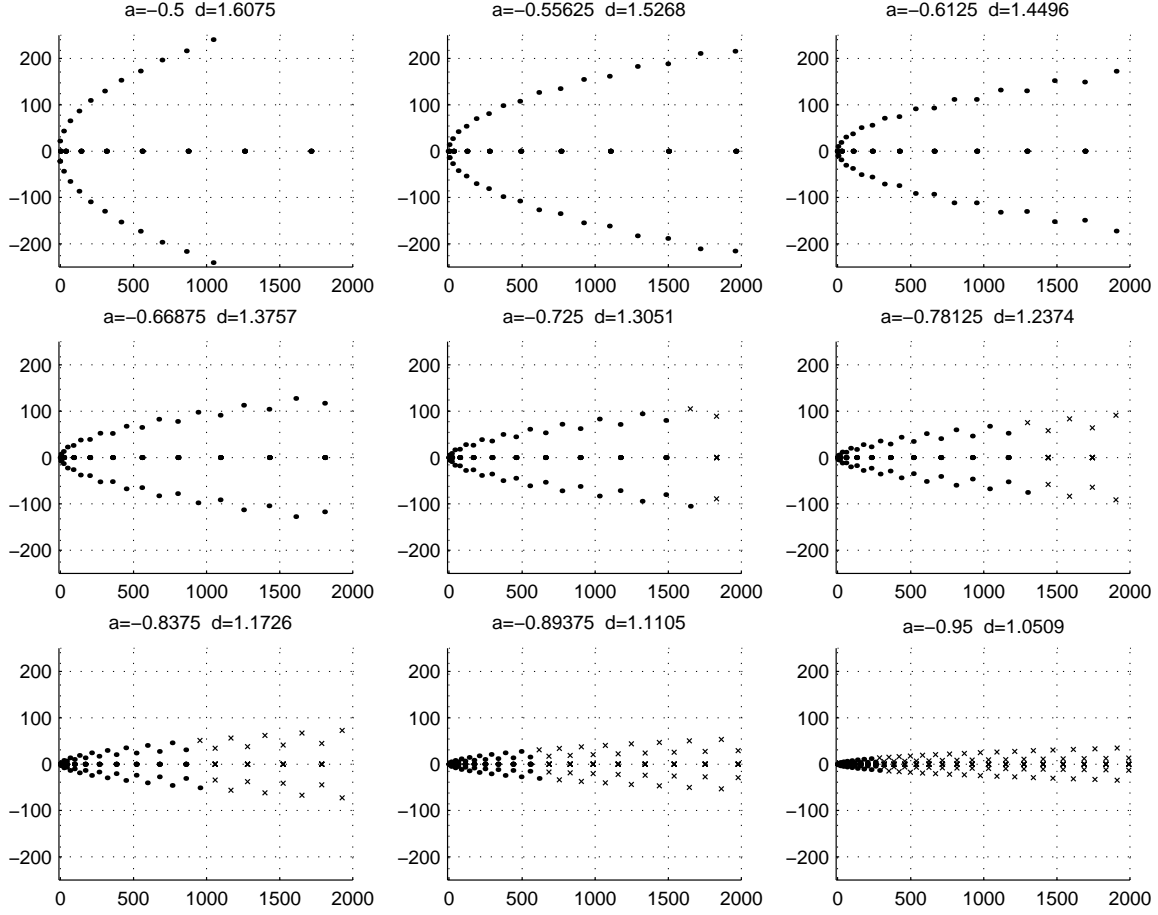
FIGURE 6. Evolution of the first 100 eigenvalues of $A_4D$ as $(a,d) \to (-1,1) \in R_6$ on $\Lambda_+(2)$. The dots are the first 100 eigenvalues while the crosses the remaining spectrum.